NATIONAL ACADEMY

DHARMAPURI

TRB MATHEMATICS

DIFFERENTAL EQUATIONS

Class will be started on 16.06.2019

'Material Available with Question papers'

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Unit-VIII - Differential Equations

Linear differential equation - constant co-efficients - Existence of solutions – Wrongskian - independence of solutions - Initial value problems for second order equations - Integration in series - Bessel's equation - Legendre and Hermite Polynomials - elementary properties - Total differential equations - first order partial differential equation - Charpits method

CLASS: IV

DIFFERENTIAL EQUATIONS

Bessel's equation

The differential equation of Bessel's equation of the form, $\mathbf{x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx}+(x^2-n^2)\, y = 0}$ is called the Bessel equation of order n (or) $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx}+(1-\frac{n2}{x^2})\, y = 0$

Bessel's fuction

The solution of Bessel's equation are called Bessel's function

Bessel function of the first kind of order n is denoted by Jn(x) defined as

$$\mathbf{J_n(x) = \sum_{r=0}^{\infty}(-1)^r \frac{1}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{2r+n}}, \text{ where } \Gamma(n+r+1) = (n+r)!$$

$$J_n(x) = \sum_{r=0}^{\infty}(-1)^r \frac{1}{r!\,(n+r)!}\left(\frac{x}{2}\right)^{2r+n}$$

The Bessel's function of order – n is,

$$J_{-n}(x) = \sum_{r=0}^{\infty}(-1)^r \frac{1}{r!\,\Gamma(-n+r+1)}\left(\frac{x}{2}\right)^{2r-n} \quad \text{(OR)}$$

$$J_{-n}(x) = \sum_{r=0}^{\infty}(-1)^r \frac{1}{r!\,(r-n)!}\left(\frac{x}{2}\right)^{2r-n}$$

The Bessel's function of order zero is,

$$J_0(x) = \sum_{r=0}^{\infty}(-1)^r \frac{1}{r!\,\Gamma(r+1)}\left(\frac{x}{2}\right)^{2r} \quad \text{(or)}$$

$$J_0(x) = \sum_{r=0}^{\infty}(-1)^r \frac{1}{r!}\left(\frac{x}{2}\right)^{2r}$$

$$\mathbf{J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 . 4^2 . 6^2} + \cdots \quad + \infty}$$

When x = 0 ,then Bessel's equation of order zero is, $\mathbf{J_0(x) = 1}$

The Bessel's function of order 1 is,

$$J_1(x) = \sum_{r=0}^{\infty}(-1)^r \frac{1}{r!\,\Gamma(1+r+1)}\left(\frac{x}{2}\right)^{2r+1}$$

$$= \sum_{r=0}^{\infty}(-1)^r \frac{1}{r!\,\Gamma(r+2)}\left(\frac{x}{2}\right)^{2r+1}$$

$$\mathbf{J_1(x) \quad = \frac{x}{2} - \frac{1}{1!2!}\left(\frac{x}{2}\right)^3 + \frac{1}{2!3!}\left(\frac{x}{2}\right)^5 - \cdots \quad + \infty}$$

Note:

1. $\Gamma\, n = \int_0^{\infty} e^{-x} x^{n-1} dx$
2. $\Gamma\, 1 = 1$
3. $\Gamma\,(n+1) = n\, \Gamma\, n$
4. $\Gamma\,(\frac{1}{2}) = \sqrt{\pi}$

Relation between $J_n(x)$ and $J_{-n}(x)$

1. J-n(x) = (-1)nJn(x), where n is positive integer
2. Jn(x) = (-1)nJn(x) ,where n is any integer
3. $J_{\frac{-1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\; cosx$
4. $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\; sinx$
5. $[J_{\frac{-1}{2}}(x)]^2 + [J_{\frac{1}{2}}(x)]^2 = \frac{2}{\pi x}$
6. $\tan x = \frac{J_{\frac{1}{2}}(x)}{J_{\frac{-1}{2}}(x)}$

7. $\cot x = \frac{J_{-\frac{1}{2}}(x)}{J_{\frac{1}{2}}(x)}$

Recurrence formula for $J_n(x)$

1. $\frac{d}{dx}\{x^n J_n(x)\} = x^n J_{n-1}(x)$
2. $\frac{d}{dx}\{x^{-n} J_n(x)\} = - x^{-n} J_{n+1}(x)$
3. xJ_n^1 -= $-nJ_n + xJ_{n-1}$
4. xJ_n^1 -= $nJ_n - xJ_{n-1}$
5. $J_{n-1} - J_{n+1} = 2J_n^1$
6. $J_{n-1}(x) + J_{n+1}(x) = (\frac{2n}{x})J_n(x)$ (or) $xJ_{n+1}(x) + xJ_{n-1}(x) = 2nJ_n(x)$

Generating function for the Bessel's function $J_n(x)$

$\exp\{\frac{1}{2}x(t - \frac{1}{t})\} = \sum_{r=-\infty}^{\infty} t^r J_n(x)$, the coefficient of t^r in the expansion of $e^{\frac{x}{2}(t-\frac{1}{t})}$ is $J_n(x)$

Legendre polynomials

The legendre differential equation is of the form

$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$ (OR) $\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\}y + n(n+1)y = 0$

$x = \infty$ is a regular singular point of legendre differential equation.

Legendre polynomial

The polynomial solution P_n of degree n of $L(x) = 0$ satisfying $P_n(1) = 1$ is called the n^th legendre polynomials.

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

The polynomial of order zero is,

$$P_0(x) = \frac{1}{2^0 0!}\frac{d^0}{dx^0}(x^2-1)^0 = 1$$

$P_0(x) = 1$

The polynomial of order 1 is,

$$P_1(x) = \frac{1}{2^1 1!}\frac{d^1}{dx^1}(x^2-1)^1 = \frac{1}{2}\frac{d}{dx}[x^2-1]$$

$$\mathbf{P_1(x) = 2x}$$

The polynomial of order 2 is,

$$P_2(x) = \frac{1}{2^2 2!}\frac{d^2}{dx^2}(x^2-1)^2$$

$$= \frac{1}{4\times 2}\frac{d^2}{dx^2}[x^4-2x^2+1] = \frac{1}{2}\frac{d}{dx}[x^3-x]$$

$$\mathbf{P_2(x) = \frac{1}{2}[3x^2-1]}$$

The polynomial of order 3 is,

$$P_3(x) = \frac{1}{2^3 3!}\frac{d^3}{dx^3}(x^2-1)^3 = \frac{1}{8\times 6}\frac{d^3}{dx^3}[x^6-3x^4+3x^2-1]$$

$$\mathbf{P_3(x) = \frac{1}{2}[5x^3-3x]}$$

The polynomial of order 4 is,

$$P_4(x) = \frac{1}{2^4 4!}\frac{d^4}{dx^4}(x^2-1)^4$$

$$\mathbf{P_4(x) = \frac{35x^4-30x^2+3}{8}}$$

Generating function for legendre polynomials

$(1-2xz+z^2)^{-\frac{1}{2}}$ is called the generating function for legentre polynomial $P_n(x)$

$(1-2xz+z^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} z^n P_n(x)$,Where $P_1(x), p_2(x), P_3(x), \ldots, P_n(x), \ldots$ are coefficient of $z, z^2, z^3, \ldots, z^n, \ldots$ in the expansion of $(1-2xz+z^2)^{-\frac{1}{2}}$

The coefficient of z^n in the polynomial Pn(x) is $\frac{(2n)!}{2^n(n!)^2}$

If Pn(x) is legendre polynomial, then

1. $P_n(1) = 1$
2. $P_n(-1) = (-1)^n$
3. $P_n(-x) = (-1)^n Pn(x)$
4. $P_n^1(1) = \frac{1}{2} n(n+1)$
5. $P_n^1(-1) = (-1)^n \frac{1}{2} n(n+1)$

The generating function formula is,

$$(1-2xz+z^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} z^n P_n(x)$$

put x =1

$$(1-2z+z^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} z^n P_n(1)$$

$$1+z+z^2+z^3+\dots +z^n+\dots = \sum_{n=0}^{\infty} z^n P_n(1)$$

Equating the coefficient of z^n on both sides, $\mathbf{P_n(1) = 1}$

put x = -1

$$(1+2z+z^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} z^n P_n(-1)$$

$$1-z+z^2-z^3+\dots (-1)^n z^n+\dots = \sum_{n=0}^{\infty} z^n P_n(-1)$$

Equating the coefficient of z^n on both sides, $\mathbf{P_n(-1) = (-1)^n}$

Replacing x by -x

$$(1+2xz+z^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} z^n P_n(-x) \text{ - - - - - - - - } (1)$$

Replacing z by –z

$$(1+2xz+z^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} (-z)^n P_n(x)$$

$$= \sum_{n=0}^{\infty} (-1)^n (z)^n P_n(x) \text{ - - - - - - - } (2)$$

Equating the coefficient of z^n on both sides (1) and (2),

$P_n(-x) = (-1)^n Pn(x)$

Since Pn(x) satisfies legendre equation

$$(1-x^2)\ P_n^{11}(x) - 2xP_n^1\ (x)\ \frac{dy}{dx} + n(n+1)\ P_n(x) = 0$$

put x = 1, $P_n(1) = 1$.

$-2P_n^1\ (1)\ + n(n+1)P_n(1) = 0$

$-2P_n^1\ (1)\ + n(n+1) = 0$

$$\boldsymbol{P_n^1\ (1) = \frac{n(n+1)}{2}}$$

Put x = -1

$-2P_n^1\ (-1)\ + n(n+1)P_n(-1) = 0 \qquad \{\ P_n(-1) = (-1)^n$

$$P_n^1\ (-1) = (-1)^{n-1}\ \frac{n(n+1)}{2}$$

Orthogonal properties

If Pn(x) is legendre polynomial, then

- $\int_{-1}^{1} P_m(x)\ P_n(x)dx = 0$, If $m \neq n$
- $\int_{-1}^{1} [\,Pn\ (x)]^2 dx = \frac{2}{2n+1}$,If m= n is called normalized legendre polynomial.