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UNIT-VIII

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UNIT-VIII

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Analytic Functions

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A **complex number** is a number of the form $a + bi$ where a, b are real numbers and i is the square root of -1. They have the algebraic structure of a field. In engineering and physics, complex numbers are used extensively to describe electric circuits and electromagnetic waves. The number i appears explicitly in the Schrödinger wave equation, which is fundamental to the quantum theory of the atom. Complex analysis, which combines complex numbers with ideas from calculus, has been widely applied to various subjects.

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Historically, complex numbers arose in the search for solutions to equations such as $x^2 = -1$. Because there is no real number x for which the square is -1, early mathematicians believed this equation had no solution. However, by the middle of the 16^{th} century, Italian mathematician Gerolamo Cardano and his contemporaries were experimenting with solutions to equations that involved the square roots of negative numbers. Swiss mathematician Leonhard Euler introduced the modern symbol i for $\sqrt{-1}$ in 1777 and expressed the famous relationship $e^{i\pi} = -1$, which connects four of the fundamental numbers of mathematics. In his doctoral dissertation in 1799, German mathematician Carl Friedrich Gauss

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proved the fundamental theorem of algebra, which states that every polynomial with complex coefficients has a complex root. The study of complex functions was continued by French mathematician Augustin Louis Cauchy, who in 1825 generalized the real definite integral of calculus to functions of a complex variable.

We first discuss about complex functions and then define the concepts limit, continuity, differentiability of complex functions. We will study in detail about ***analytic functions***, an important class of complex functions, which plays a central role in complex analysis.

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Regions in the Complex Plane

In this section, we recollect some facts concerned with sets of complex numbers, or points in the z plane, and their closeness to one another.

For any $\varepsilon > 0$, an ε- ***neighborhood*** of a given point z_0 is the set $|z - z_0| < \varepsilon$. It consists of all points z lying inside but not on a circle centered at z_0 and with positive radius ε.

A ***deleted neighborhood*** of z_0, or ***punctured disk***, $0 < |z - z_0| < \varepsilon$ consisting of all points z in an ε- neighborhood of z_0 except for the point z_0 itself.

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A point z_0 is said to be an **interior point** of a set S whenever there is some neighborhood of z_0 that contains only points of S. z_0 is called an ***exterior point*** of S when there exists a neighborhood of it containing no points of S. If z_0 is neither of these, it is a ***boundary point*** of S. Thus, a boundary point is a point all of whose neighborhoods contain at least one point in S and at least one point not in S. The totality of all boundary points is called the ***boundary*** of S.

A set is called ***open*** if it contains none of its boundary points. Clearly, a set is open if and only if each of its points is an interior point. A set is ***closed*** if it contains all of its boundary points, and the ***closure*** of a set S is the closed set consisting of all points in S together with the boundary of S.

An open set S is said to be ***connected*** if each pair of points z_1 and z_2 in S can be joined by a polygonal line, consisting of a finite number of line segments joined end to end, that lies entirely in S.

A nonempty open set that is connected is called a ***domain***. A domain together with some, none, or all of its boundary points is said to be a ***region***.

A set S is ***bounded*** if every point of S lies inside some circle $|z| = R$; otherwise, it is ***unbounded***. A point z_0 is said to be an ***accumulation point*** of a set S if each deleted neighborhood of z_0 contains at least one point of S.

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Functions of a Complex Variable

Let S be a set of complex numbers. A ***function*** f defined on S is a rule that assigns to each z in S a complex number w. The number w is called the ***value*** of f at z and is denoted by $f(z)$. i.e., $w = f(z)$. The set S is called the ***domain of definition*** of f.

Let $w = f(z)$ be a complex function of the complex variable z. Let $w = u+iv$ and $z = x + iy$. Then, u and v depends upon the values of the real variables x and y. Therefore, $f(z) = u(x, y)+iv(x, y)$. This shows that any complex function $f(z)$ of a complex variable $z = x + iy$ is equivalent to a pair of two real-valued functions u and v of the real variables x and y.

In polar coordinates, $z = re^{i\theta}$, we have $f(z) = u(r, \theta) + iv(r, \theta)$.

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Example 1.

Consider the complex function $f(z) = z^2$, then $f(x+iy) = (x+iy)^2 = x^2 - y^2 + i2xy$. Hence $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$. When polar coordinates are used, $f(re^{i\theta}) = (re^{i\theta})^2 = r^2e^{i2\theta} = r^2cos2\theta + ir^2sin2\theta$.. Therefore, $u(r,\theta) = r^2cos2\theta$ and $v(r,\theta) = r^2sin2\theta$.

If n is zero or a positive integer and if $a_0, a_1, a_2, ..., a_n$ are complex constants, where $a_n \neq 0$, the function $P(z) = a_0 + a_1z + a_2z^2 + ... + a_nz^n$ is called a ***polynomial*** of ***degree*** n. Here, the sun has only a finite number of terms and the domain of definition is the entire z plane. A quotient of the form $\frac{P(z)}{Q(z)}$ where, $P(z)$ and $Q(z)$ are polynomials is called a ***rational function*** and is defined at each point z where $Q(z) \neq 0$.

Problem 1. SRIMAAN

Express the function $f(z) = z^3 + z + 1$ in the form $f(z) = u(x,y) + iv(x,y)$.

Solution.

Let $z = x+iy$. Then, $f(z) = z^3+z+1 = (x+iy)^3+(x+iy)+1$. On simplification, we get $f(z) = (x^3 - 3xy^2 + x + 1) + i(3x^2y - y^3 + y)$.

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Limit of a Function of a Complex Variable

Let a function f be defined at all points z in some deleted neighborhood z_0. Then we say that the ***limit*** of $f(z)$ as z approaches z_0 is a number w_0, or in symbols $lim_{z\to z_0}f(z) = w_0$, if for each positive number ε, there is a positive number δ such that $|f(z) - w_0| < \varepsilon$ whenever $0 < |z - z_0| < \delta$. (i.e., the limit of $f(z)$ as z approaches z_0 is the number w_0, if the point $w = f(z)$ can be made arbitrarily close to w_0 if we choose the point z close enough to z_0 but distinct from it.) Geometrically, this means that for each ε-neighborhood $|w - w_0| < \varepsilon$ of w_0, there exists a deleted neighborhood $0 < |z - z_0| < \delta$ of z_0 such that every point z in it has an image w lying in the ε-neighborhood.

Note that limit of a function $f(z)$ at a point z_0 is unique, if it exists.

Theorem SRIMAAN

Suppose that $f(z) = u(x, y) + iv(x, y)$, $(z = x + iy)$ *and* $z_0 = x_0 + iy_0, w_0 = u_0 + iv_0$. *Then,* $lim_{z \to z_0} f(z) = w_0$ *if and only if* $lim_{(x,y) \to (x_0,y_0)} u(x, y) = u_0$ *and* $lim_{(x,y) \to (x_0,y_0)} v(x, y) = v_0$.

Similar rules as in the case of the limits of real functions, will hold in the complex case. For instance, if $lim_{z \to z_0} f(z) = w_0$ and if $lim_{z \to z_0} F(z) = W_0$, then

$$lim_{z \to z_0}[f(z) + F(z)] = w_0 + W_0,$$

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$$lim_{z \to z_0}[f(z)F(z)] = w_0 W_0,$$

and, if $W_0 \neq 0$,

$$lim_{z \to z_0} \frac{f(z)}{F(z)} = \frac{w_0}{W_0}.$$

Also, $lim_{z \to z_0} c = c$ and $lim_{z \to z_0} z = z_0$, where z_0 and c are any complex numbers, and $lim_{z \to z_0} z^n = z_0^n$ $(n = 1, 2, ...)$.

The limit of a polynomial $P(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n$ as z approaches a point z_0 is the value of the polynomial at that point. i.e., $lim_{z \to z_0} P(z) = P(z_0)$.

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Limits involving the Point at Infinity

The complex plane together with the *point at infinity* is called the ***extended complex plane***. To visualize the point at infinity, denoted by ∞ one can think of the complex plane as passing through the equator of a unit sphere centered at the origin. To each point z in the plane there corresponds exactly one point P on the surface of the sphere. The point P is the point where the line through z and the north pole N intersects the sphere. In like manner, to each point P on the surface of the sphere, other than the north pole N, there corresponds exactly one point z in the plane. By letting the point N of the sphere correspond to the point at infinity, we obtain a *one to one correspondence* between the points of the sphere and the points of the extended complex plane. The sphere is known as the ***Riemann sphere***, and the correspondence is called a ***stereographic projection***.

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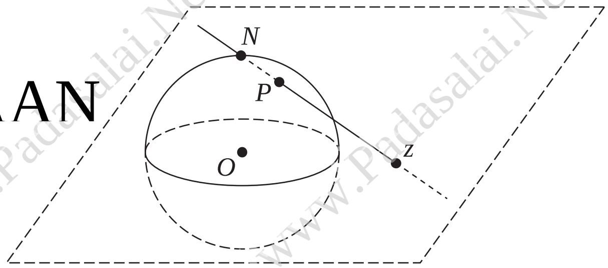

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Note that in the above identification, the exterior of the unit circle centered at the origin in the complex plane corresponds to the upper hemisphere with the equator and the point N deleted. Moreover, for each small positive number $\varepsilon > 0$, those points in the complex plane exterior to the circle $|z| = \dfrac{1}{\varepsilon}$ correspond to points on the sphere close to N. Therefore, the set $|z| = \dfrac{1}{\varepsilon}$ is called an ε-neighborhood of ∞. With this definition of ε-neighborhood of ∞, we can define

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limits involving the point at infinity as in 1.2.1. Also, we have: If z_0 and w_0 are points in the z and w planes, respectively, then $lim_{z\to z_0}f(z) = \infty$ if and only if $lim_{z\to z_0}\dfrac{1}{f(z)} = 0$ and $lim_{z\to\infty}f(z) = w_0$ if and only if $lim_{z\to 0}f(\dfrac{1}{z}) = w_0$. Moreover, $lim_{z\to\infty}f(z) = \infty$ if and only if $lim_{z\to 0}\dfrac{1}{f(1/z)} = 0$.

Continuity

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A function f is said to be ***continuous*** at a point z_0 if: (i) $lim_{z\to z_0}f(z)$ exists, (ii) $f(z_0)$ exists, and (iii) $lim_{z\to z_0}f(z) = f(z_0)$.

A function of a complex variable is said to be ***continuous*** in a region R if it is continuous at each point in R.

Note that if two functions are continuous at a point, then their sum and product are also continuous at that point and their quotient is continuous at any such point if the denominator is not zero there. Also, a composition of continuous functions is itself continuous. If $f(z) = u(x, y) + iv(x, y)$, then the function $f(z)$ is continuous at a point $z_0 = (x_0, y_0)$ if and only if its component functions $u(x, y)$ and $v(x, y)$ are continuous at (x_0, y_0).

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Theorem

If a function $f(z)$ is continuous and nonzero at a point z_0, then $f(z) \neq 0$ throughout some neighborhood of that point.

Proof.

Assume that $f(z)$ is continuous and nonzero at z_0. Let $\varepsilon = \dfrac{|f(z_0)|}{2}$. Then corresponding to this $\varepsilon > 0$, there is a positive number δ such that $|f(z) - f(z_0)| < \dfrac{|f(z_0)|}{2}$ whenever $|z - z_0| < \delta$. Now, if there is a point z in the neighborhood

$|z - z_0| < \delta$ at which $f(z) = 0$, then we have $|f(z_0)| < \frac{|f(z_0)|}{2}$, which is a contradiction. This shows that $f(z) \neq 0$ throughout the neighborhood $|z-z_0| < \delta$ of z_0.

Theorem

If a function f is continuous throughout a region R that is both closed and bounded, there exists a nonnegative real number M such that $|f(z)| \leq M$ for all points z in R, where equality holds for at least one such z.

Derivatives

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Let f be a function whose domain of definition contains a neighborhood $|z-z_0| < \delta$ of a point z_0. The ***derivative*** of f at z_0 is the limit

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$$f'(z_0) = lim_{z\to z_0}\frac{f(z) - f(z_0)}{z - z_0},$$

and the function f is said to be ***differentiable*** at z_0 when $f'(z_0)$ exists.

Writing $\triangle z = z - z_0$, where $z \neq z_0$, we have

$$f'(z_0) = lim_{\triangle z\to 0}\frac{f(z_0 + \triangle z) - f(z_0)}{\triangle z}.$$

Since f is defined throughout a neighborhood of z_0, the number $f(z_0 + \triangle z)$ is always defined for $|\triangle z|$ sufficiently small. If we write $w = f(z + \triangle z) - f(z)$, then

$$f'(z) = \frac{dw}{dz} = lim_{\triangle z\to 0}\frac{\triangle w}{\triangle z}.$$

2. Show that

   (a) $lim_{z \to i} \dfrac{iz^3 - 1}{z + i} = 0.$

   (b) $lim_{z \to z_0} z^n = z_0^n \quad (n = 1, 2, 3, ...)$(Hint: Use mathematical induction.)

   (c) $lim_{z \to \infty} \dfrac{4z^2}{(z-1)^2} = 4.$

   (d) $lim_{z \to \infty} \dfrac{z^2 + 1}{z - 1} = \infty.$

3. Show that the limit of the function $f(z) = (\frac{z}{\bar{z}})^2$ as z tends to 0 does not exist.

4. Show that $lim_{z \to z_0} f(z)g(z) = 0$ if $lim_{z \to z_0} f(z) = 0$ and if there exists a positive number M such that $|g(z)| \leq M$ for all z in some neighborhood of z_0.

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5. Show that a set S is unbounded if and only if every neighborhood of the point at infinity contains at least one point in S.

6. Find $f'(z)$, where (a). $f(z) = 3z^2 - 2z + 4$ (b). $f(z) = \dfrac{z+1}{2z-1}$ $(z \neq \frac{1}{2})$

7. Show that $f'(z)$ does not exist at any point z when (a). $f(z) = Rez$, (b). $f(z) = Imz$.

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Cauchy - Riemann Equations

In this section, we obtain a pair of equations that the first-order partial derivatives of the component functions u and v of a function $f(z) = u(x, y) + i\,v(x, y)$ must satisfy at a point $z_0 = (x_0, y_0)$ when the derivative of f exists there. We also derive an expression for $f'(z_0)$ in terms of partial derivatives of u and v.

Therefore, we must have $u_x(x_0, y_0) = v_y(x_0, y_0)$ and $u_y(x_0, y_0) = -v_x(x_0, y_0)$. Also, we have $f'(z_0) = lim_{\triangle z \to 0}\dfrac{\triangle w}{\triangle z} = u_x(x_0, y_0) + i\ v_x(x_0, y_0)$.

Example

Let $f(z) = |z|^2 = u(x, y) + i\ v(x, y)$. Then, we have $u(x, y) = x^2 + y^2$ and $v(x, y) = 0$.

Then $u_x = 2x$, $u_y = 2y$, and $v_x = v_y = 0$. So the Cauchy - Riemann equations holds at a point (x, y) only if $2x = 0$ and $2y = 0$, i.e., only if $x = y = 0$. Therefore, $f'(z)$ does not exist at any nonzero point.

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From, Exercise 1 given below, we know that satisfaction of the Cauchy-Riemann equations at a point is not sufficient to ensure the existence of the derivative of a function at that point.

Now we derive a sufficient condition for differentiability of $f(z)$ at a point $z_0 = x_0 + i\ y_0$.

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For proving this theorem we make use of the following result from your *Vector Calculus* (V- Semester) Course.

The increment Theorem for Functions of Two Variables:

Suppose that the first order partial derivatives of $f(x, y)$ *are defined throughout an open region* R *containing the point* (x_0, y_0) *and that* f_x *and* f_y *are continuous at* (x_0, y_0) *. Then the change,* $\triangle z = f(x_0+\triangle x, y_0+\triangle y)-f(x_0, y_0)$ *in the value of* f *that results from moving from* (x_0, y_0) *to another point* $x_0 + \triangle x, y_0 + \triangle y)$ *in* R *satisfies an equation of the form* $\triangle z = f_x(x_0, y_0)\triangle x+f_y(x_0, y_0)\triangle y+\varepsilon_1\triangle x+\varepsilon_2\triangle y$ *in which* $\varepsilon_1, \varepsilon_2 \to 0$ *as* $\triangle x, \triangle y \to 0$.

Theorem

Let the function $f(z) = u(x, y) + i\ v(x, y)$ *be defined throughout some* ε-

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