SRIMAAN

GOVT.POLYTECHNIC-TRB

MATHEMATICS

UNIT-I&III

ALGEBRA

REAL ANALYSIS

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ANNIHILATING POLYNOMIAL:

The annihilator of a set $S \subseteq V$ is $Sa = \{f \in V^* \mid f(v) = 0, \forall v \in S\}$.

Definition:

Let V be a vector space over the field F, where $F = \Re$ (or) $F = C$. An **inner product** on V is a function $\langle \cdot,\cdot \rangle : V \times V \to F$ such that for all u,v,w ∈ V and a,b ∈ F, the following hold:

(i) $\langle v,v\rangle \geq 0$ and $\langle v,v\rangle = 0$ iff $v=0$.
(ii) $\langle au+bv,w\rangle = a\langle u,w\rangle + b\langle v,w\rangle$.
(iii) For $F = \Re$: $\langle u,v\rangle = \langle v,u\rangle$;
For $F = \Re$ C: $\langle u,v\rangle = \langle v,u\rangle$ (where bar denotes complex conjugation).

A real (or complex) **inner product space** is a vector spaceV over $\Re$ (or C), together with an inner product defined on it. In an **inner product space** V, the norm, orlength, of a vector v∈ V is $\|v\| = \sqrt{(v,v)}$. A vectorv∈ V is a **unit vector** if $\|v\|=1$.

The angle between two nonzero vectors u and v in a real inner product space is the real number θ, $0 \leq \theta \leq \pi$, such that $\langle u,v\rangle = \|u\|\,\|v|\cos\theta$.
Let V be an inner product space. The **distance** between two vectors u and v is $d(u,v)=\|u-v\|$.

Gram–Schmidt Orthogonalization process:

Definition:

Let $\{a_1,a_2,\ldots,a_n\}$ be a basis for a subspace S of an inner product space V. An orthonormal basis $\{u_1,u_2,\ldots,u_n\}$ for S can be constructed using the following Gram–Schmidt orthogonalization process:

$u_1 = a_1 / \| a_1 \|$ and $u_k = a_k - \sum_{i=1}^{k-1} \langle a_k, u_i \rangle u_i \,/\, \| a_k - \sum_{i=1}^{k-1} \langle a_k, u_i \rangle u_i \|$ for k=2,…..,n.

Jordan Canonical Form:

A Jordan canonical form of matrix A, denoted JA (or) JCF(A), is a Jordan matrix that is similar to A. It is conventional to group the blocks for the same eigen value together and to order the Jordan blocks with the same eigen value in non-increasing size order.

Jordan canonical form:

Let $A \in C_{nxn}$ have the Jordan canonical form $Z^{-1} AZ = J_A$ = diag $(J_1 \ \lambda_1), J_2(\lambda_2), \ldots, J_p(\lambda_p)$, where Z is non singular.

$$j_k(\lambda_k) = \begin{bmatrix} \lambda_k & 1 & & \\ & \lambda_k & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_k \end{bmatrix} \in C^{m_k x m_k} \text{ and } m_1 + m_2 + \cdots + m_p = n.$$

The **Jordan in variants** of A are the following parameters:

- The set of distinct eigenvalues of A.
- For each eigenvalue λ, the number b_λ and sizes $p_1, \ldots, p_{b\lambda}$ of the Jordan blocks with eigenvalue λ in a Jordan canonical form of A.

Schur Complements:

The Schur complement of A_{11} in A is the matrix $A_{22} - A_{21} A_{11}^{-1} A_{12}$, sometimes denoted A/A_{11}.

Direct Sum Decompositions:

The sum of subspaces W_i, for i = 1,...,k, is k $_i$=1 $W_i = W_1 + \cdots + W_k = \{w_1 + \cdots + w_k \mid w_i \in W_i\}$.

The sum $W_1 + \cdots + W_k$ is a direct sum if for all i = 1,...,k, we have $W_i \cap W_j$ =i Wj ={0}. $W = W_1 \oplus \cdots \oplus W_k$ denotes that $W = W_1 + \cdots + W_k$ and the sum is direct. The subspaces W_i, for i =i,...,k, are independent iff (or) $w_i \in W_i$, $w_1 + \cdots + w_k = 0$ implies $w_i = 0$ for all i =1,...,k. Let V_i, for i =1,...,k, be vector spaces over F.

Inner Product Spaces:

Let V be a real vector space. Suppose to each pair of vectors u,v ε V there is assigned a real number, denoted by $\langle u, v\rangle$. This function is called a (real) inner product on V if it satisfies the following axioms:

1. (**Linear Property**): $\langle au_1 + bu_2, v\rangle = a\langle u_1, v\rangle + b\, b\langle u_2, v\rangle$.
2. (**Symmetric Property**): $\langle u, v\rangle = \langle v, u\rangle$
3. (**Positive Definite Property**): $\langle u, u\rangle \geq 0$ & $\langle u, u\rangle = 0$ iff u=0.

The vector space V with an inner product is called a **(real) inner product space.**

Example of Inner Product Spaces:

1.Let u=(1,3,-4,2), v =(4,-2,2,1), w =(5,-1,-2,6) in R^4. Show that <3u-2v,w>=3<u,w>-2<v,w>.

Sol.

By definition, <u,w>=5-3+8+12=22 and <v,w>=20+2-4+6=24.
Note that, 3u-2v=(-5,13,-16,4).
Thus, <3u-2v,w>=-25-13+32+24=18.

As expected, $3<u,w>-2<v,w>=3(22)-2(24)=18$.

There fore, $3<u,w>-2<v,w>=<3u-2v,w>$.

Cayley–Hamilton Theorem

Every matrix A is a root of its characteristic polynomial.

(or)

Every square matrix satisfies its characteristic equation.

Invariant Direct-Sum Decompositions:

A vector space V is termed the direct sum of subspaces $W_1,\ldots,W_r$, written $V = W_1 \oplus W_2 \oplus,\ldots \oplus W_r$ if every vector $v \in V$ can be written uniquely in the form $v = w_1 + w_2 + \ldots + w_r$, with $w_i \in W_i$.

Primary Decomposition Theorem:

Let $T:V_1 \rightarrow V$ be a linear operator with minimal polynomial $m(t)=f_1(t)^{n1} f_2(t)^{n2},\ldots f_r(t)^{nr}$ where the $f_i(t)$ are distinct monic irreducible polynomials.

Then V is the direct sum of T-invariant subspaces $W_1,\ldots,W_r$, where W_i is the kernel of $f_i(T)^{ni}$. Moreover, $f_i(t)^{ni}$ is the minimal polynomial of the restriction of T to W_i.

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GOVT.POLYTECHNIC COLLEGE-LECTURER

MATHEMATICS

UNIT-I

REAL ANALYSIS

Discontinuities:

1. First kind: (infinite) :

$f(x^+) \neq f(x^-)$ **(ie.,)** $f(x^+) \,\&\, f(x^-)$ exists but not equal (or) not both finite.

2. Second kind: (Jump discontinuity):

Either $f(x^+)(or) f(x^-)$ does not exists(or) both finite.

3. Third kind: (Removable or Point discontinuity)

$f(x^+) = f(x^-) \neq f(x)$ **(or)** $\lim_{x\to a} f(x) \neq f(a)$ **finite.**

Example:

1. If $f(x) = \frac{\sin x}{x}, x \neq 0, f(0) = 37,$ then f has a removable discontinuity at x=0.

2. If $f(x) = 0, x < 0, f(x) = 1, x > 0, f(0) = 175,$ then f has a Jump discontinuity at x=0.

3. If $f(x) = 0, x < 0, f(x) = \frac{1}{x}, x > 0,$ then f has an infinite discontinuity at x=0.

Theorem:

Let f:R→R ,and assume f monotone. Then all discontinuities of f are jumps and f has **at most countably many discontinuities**.

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REAL ANALYSIS PROBLEMS

1. Explain why each of the following sequences converges and in the case of (i) and (ii) determine the limits $x_n = \frac{103n^2 - 8}{4n^2 + 99n - 3}$.

Sol. The sequence converges because it is a combination of standard convergent sequences.

We have $x_n = \frac{103n^2 - 8}{4n^2 + 99n - 3}$

$$= \frac{103 - 8/n^2}{4 + 99/n - 3/n^2}$$

→ 103 4 as n → ∞.

2. Determine $x_n = -n + \sqrt{n^2 + 3n}$

Sol. $x_n = -n + \sqrt{n^2 + 3n}$

$$= \frac{(n^2 + 3n) - n^2}{\sqrt{(n^2 + 3n)} + n} \Rightarrow \frac{3n}{n\sqrt{(1 + 3/n)} + n} \Rightarrow \frac{3n}{\sqrt{(1 + 3/n)} + 1} \to \frac{3}{2} \, as \; n \to \infty$$.

3. Determine the limits $x_n = \sum_{k=1}^{n} \frac{3k^2 + 2k}{2^k}$.

Sol. (x_n) is a sequence of partial sums (i.e. a series). With $a_k = (3k^2 + 2k)/2^k$ the ratio test gives

$$\frac{a_{k+1}}{a_k} = \left(\frac{1}{2}\right)\frac{3(k+1)^2 + 2(k+1)}{3k^2 + 2k}$$

$$= \left(\frac{1}{2}\right)\frac{3(1+1/k)^2 + 2(1/k + 1/k^2)}{3 + 2/k} \to \frac{1}{2} as k \to \infty.$$

4. Define what it means for the sequence (x_n) to be a Cauchy sequence?

Sol (x_n) is a Cauchy sequence if for every $\in\ > 0$ there exists a N such that

$|x_n - x_m| < \in$ or all n,m satisfying $n \geq N$ and $m \geq N$.

5. If $a_k = \frac{1}{k2^k}, b = \frac{1}{2^k}, s_n = \sum_{k=1}^{n} a_k$ and $t_n = \sum_{k=1}^{n} b_k$.

Find the limits of the sequences (a_{k+1}/a_k) and (b_{k+1}/b_k).

Sol. Let $\frac{a_{k+1}}{a_k} = \frac{k2^k}{(k+1)2^{(k+1)}} = \frac{k}{2(k+1)} = \frac{1}{2(1+1/k)} \to \frac{1}{2} as\ k \to \infty$

$$\frac{b_{k+1}}{b_k} = \frac{(k+1)2^k}{k2^{(k+1)}} = \frac{k+1}{2k} = \frac{1+1/k}{2} \to \frac{1}{2} as\ k \to \infty.$$

6. Determine the limits of the following sequences (x_n) whose nth term x_n is given below. $x_n = \frac{7n^4 + n^2 - 2}{14n^4 + 5n - 4}$.

Sol. $$x_n = \frac{7n^4 + n^2 - 2}{14n^4 + 5n - 4} = \frac{7 + (1/n^2) - (2/n^4)}{14 + (5/n^3) - (4/n^4)} \to \frac{7}{14} = \frac{1}{2} as\ n \to \infty.$$

In the above we have used the result that 1/n → 0 as n →∞ and a result about combining convergent sequences and noting that the denominator converges to a non-zero value n→∞ .

7. Determine $x_n = \dfrac{n^3+2n^2+1}{6n^3+n+4}$.

Sol. $x_n = \dfrac{n^3+2n^2+1}{6n^3+n+4} = \dfrac{1+(2/n)+(1/n^3)}{(6+(1/n^2)+(4/n^3))} \to \dfrac{1}{6}\, as\;\; n \to \infty$.

We have used the result that 1/n → 0 as n →∞ and a result about combining convergent sequences and noting that the denominator converges to a non-zero value.

8. Determine $x_n = \dfrac{n^2+n+1}{3n^2+4}$.

Sol. $x_n = \dfrac{n^2+n+1}{3n^2+4} = \dfrac{1+(1/n)+(1/n^2)}{3+(4/n^2)} \to \dfrac{1}{3}$ as $n \to \infty$.

We have used the result that 1/n → 0 as n → ∞ and a result about combining convergent sequences. The denominator converges and it is at least 3 for all n.

Extended Real number system:

Definition:

In treating ∞and -∞ as numbers, we are extending the real number system. We have is $\mathcal{R}$U{-∞,∞}. This is called the extended real number system'

Note: It is sometimes denoted $\mathcal{R}^{\#}$.

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