NATIONAL ACADEMY

DHARMAPURI

TRB MATHEMATICS

Fourier series and Fourier Integrals

'Material Available with Question papers'

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TEST BACTH –SATURDAY & SUNDAY

NATIONAL ACADEMY DHARMAPURI

CLASS - III

The Laplace transforms

If a function f(t) is defined for all positive values of the variable t and if $\int_0^\infty e^{-st} f(t)dt$ exists .It is equal to F(s) , then F(s) is called Laplace transform of f(t) and denoted by L{f(t)}

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = F(s)$$

1. $L(e^{-st}) = \frac{1}{s+a}$
2. $L(e^{at}) = \frac{1}{s-a}$
3. L(coshat)= $\frac{s}{s^2-a^2}$
4. L(sinhat)= $\frac{a}{s^2-a^2}$
5. L(cosat)= $\int_0^\infty e^{-st} cosatdt = \frac{s}{s^2+a^2}$
6. L(sinat)= $\int_0^\infty e^{-st} sinatdt = \frac{a}{s^2+a^2}$
7. $L(t^n) = \frac{(n+1)!}{s^{n+1}}$

Complex form of the Fourier integral Formula

If F(x)is pointwise continuously differentiable in (-π, π) then f(x) has the complex Fourier series expansion,

f(x) = $\sum_{n=1}^{\infty} C_n e^{-inx}$ Where the coefficients C_n are given by

$$C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{-inx}$$

Fourier integral theorem

If f(x) is piece wise continuously differentiable and Integrable in the entire line (x axis),then,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) e^{i(t-x)s} dtds$$

This is the complex form of Fourier integral

$$f(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \{\frac{1}{\sqrt{2\pi}} f(t) e^{its} dt\} e^{-ixs} ds$$

$$f(x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} F(s) e^{-ixs} d\, s$$

Where, F(s) == $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} f(t) e^{ist}$ dt f(x) = $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-ixs}\,d\,s$

F(s) == $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{ist}\,d$t is called the Fourier transform of f(x)

Note :

1. F(s) is a complex function if f(t) is real function
2. F(s) is a real function if f(t) is real even function
3. F(s) is a purely imaginary if f(t) is real odd function
4. If F{f(x)} = F(s) ,then, f(x) = $F^{-1}\{F(s)\}$. F^{-1} denotes the inverse of F

Properties of Fourier transform

- F{af(x)+bf(x)} = aF{f(x)}+bf{f(x)}
- F{f(x-a)} = $e^{ias}F(s)$
- F{e^{iax} f(x)} = F(s+a) where F(s) = F[f(x)]
- F{f(ax)] = $\frac{1}{|a|}F(\frac{s}{a})$,a ≠ 0
- F{f(-x)} =F(-s)
- F{$x^n f(x)\} = (-1)^n \frac{d^n}{dx^n} F[f(x)]$
- F{$\frac{d^n}{dx^n} f(x)\} = (-is)^n F(s)$
- F{F(x)} = f(-s)
- F{F(-x)} = f(s)
- F{$\overline{f(x)}$,}=$\overline{F(-s)}$ where $\overline{f(x)}$ *stands for the complex conjugate of* $f(x)$
- F{$\overline{f(-x)}$, }= $\overline{F(s)}$

Fourier cosine Transform

If f(x) is an even function, it is denoted by $f_+(x)$

The Fourier cosine transform of f(x) is defined as the Fourier transformation of $f_+(x)$ and is denited by $F_c(s)$ [real part of F{$f_+(x)$}]

$F_c(s)$ = F[$f_+(x)$] = $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_+(x)\, e^{isx}\,dx$

$F_c(s) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos(sx)\,dx$

f(x) = $\sqrt{\frac{2}{\pi}}\int_0^{\infty} F_c(s)\cos(sx)\,ds$ are Fourier cosine transform pair.

Fourier sine Transform

If f(x) is an odd function, it is denoted by $f_-(x)$

The Fourier sine transform of f(x) is defined as the Fourier transformation of $f_-(x)$ and is denited by $F_s(s)$[Imaginary part of F{$f_-(x)$}]

$F_s(s)$ = F[$f_+(x)$] = $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_-(x)\, e^{isx}\,dx$

$F_s(s) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin(sx)\,dx$

f(x) = $\sqrt{\frac{2}{\pi}}\int_0^{\infty} F_s(s)\sin(sx)\,ds$ are Fourier sine transform pair.

Properties of $F_c(s)$ and $F_s(s)$

1. (a) $F_c\{af(x) + bf(x)\}$ = a$F_c\{f(x)\}$ +b$F_c\{f(x)\}$

   (b) $F_s\{af(x) + bf(x)\}$ = a$F_s\{f(x)\}$ +b$F_s\{f(x)\}$

2. $F_c\{f(x)\cos(ax)\} = \frac{1}{2}[F_c(s+a) + F_c(s-a)]$

3. $F_c\{f(x)\sin(ax)\} = \frac{1}{2}[F_s(s+a) + F_s(a-s)]$

$F_c\{f(x)\cos(ax)\} = \frac{1}{2}[F_s(s+a) - F_s(s+a)]$

$F_s\{f(x)\sin(ax)\} = \frac{1}{2}[F_c(s-a) - F_c(s+a)]$

1. $F_c\{f(ax)\} = \frac{1}{a}F_c(\frac{s}{a})$
2. $F_s\{f(ax)\} = \frac{1}{a}F_s(\frac{s}{a})$
3. $F_c\{f^1(x)\} = -\sqrt{\frac{2}{\pi}}$ f(0) +s$F_s(s)$ if f(x) → 0 as x → 0
4. $F_s\{f^1(x)\}$ = -s$F_c(s)$
5. $F_c\{f^{11}(x)\} = -\sqrt{\frac{2}{\pi}} f^1(0) - s^2 F_c(s)$
6. $F_s\{f^{11}(x)\} = \sqrt{\frac{2}{\pi}} sf(0) - s^2 F_s(s)$
7. $F_s\{F_s(x)\}$ = f(x)

Parseval's identity

$$\int_0^\infty |F_c(s)|^2 ds = \int_0^\infty |F_s(s)|^2 ds = \int_0^\infty |f(x)|^2 ds$$

Example

1. Find $F_c\{e^{-ax}\}$ and $F_c\{e^{-ax}\}$

$$F_c\{e^{-ax}\} = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos(sx) dx$$

$$= \sqrt{\frac{2}{\pi}} L\{\cos(sx)\}$$

$$= \sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2}$$

$$F_s\{e^{-ax}\} = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin(sx) dx$$

$$= \sqrt{\frac{2}{\pi}} L\{\sin(sx)\}$$

$$= \sqrt{\frac{2}{\pi}} \frac{s}{s^2+a^2}$$

2. Prove that (i) $F_c\{xf(x)\} = \frac{dF_s}{ds}$ (ii)) $F_s\{xf(x)\}$ = - $\frac{dF_c}{ds}$

(i) $F_s(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin(sx) dx$

$$\frac{d[F_s(s)]}{ds} = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial}{\partial x}[f(x)\sin(sx)]dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty xf(x)\cos(sx)dx$$

$$= F_c\{xf(x)\}$$

$(ii) F_c(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos(sx) dx$

$$\frac{d[F_c(s)]}{ds} = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial}{\partial x}[f(x)\cos(sx)]dx$$

$$= -\sqrt{\frac{2}{\pi}} \int_0^\infty xf(x)\sin(sx)dx$$

$$= -F_s\{xf(x)\}$$

$F_s\{xf(x)\} = -\frac{dF_c}{ds}$

3. $F_c\{xe^{-ax}\} = \frac{d}{ds} F_s\{e^{-ax}\}$

$$= \frac{d}{ds}\sqrt{\frac{2}{\pi}}\frac{s}{s^2+a^2}$$

$$= \sqrt{\frac{2}{\pi}}\frac{a^2-s^2}{(s^2+a^2)^2}$$

4. $F_s\{xe^{-ax}\} = -\frac{d}{ds} F_c\{e^{-ax}\}$

$$= -\sqrt{\frac{2}{\pi}}\frac{d}{ds}\frac{a}{s^2+a^2}$$

$$= \sqrt{\frac{2}{\pi}}\frac{2as}{(s^2+a^2)^2}$$

5. $F_c\left\{\frac{1}{1+x^2}\right\} = \sqrt{\frac{2}{\pi}}\, e^{-s}$
6. $F_c\left\{\frac{x}{1+x^2}\right\} = \sqrt{\frac{2}{\pi}}\, e^{-s}$

7. $F_c\left\{\frac{1}{\sqrt{x}}\right\} = F_s\left\{\frac{1}{\sqrt{x}}\right\} = \frac{1}{\sqrt{s}}$

Convolution

The convolution of two function f(x) and g(x) is difined by $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)g(x-t)dt$ and it is denoted by f*g

Convolution theorem

F(f*g) = F(s)G(s) where F(s) and G(s) are the Fourier transform of f(x) and g(x) respectively.

Identities

(i) $\int_0^{\infty} F_c(s)G_c(s)dx = \int_0^{\infty} f(x)g(x)dx$

(ii) $\int_0^{\infty} F_s(s)G_s(s)dx = \int_0^{\infty} f(x)g(x)dx$

(iii) $f(x) * g(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)g(x-t)dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(t)f(x-t)dt$

Parseval's identity

A function f(x) and its Fourier transform F(s) satisfy the identity

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{0}^{\infty} |F(s)|^2 ds$$

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