TRB – PG (MATHS)
Complete Study Material

- Unit Wise - Notes
- Unit Wise Questions
- Previous Year Question Paper (Problem Solved)
- Model Questions
- Education (Jeba’s Education Guide)
- Gk (Special Jeba’s GK guide with only full material)

Unit – VI

Functional Analysis


Banach Spaces

Definition (Normed Linear Space)

- Let $N$ be a linear spaces. A norm on $N$ is a real function $\|\|: N \rightarrow R$ satisfying the following conditions:
  
  For $x, y \in N$ and $\alpha$ a scalar,

  - $\|x\| \geq 0$ and $\|x\| = 0 \iff x = 0$
  - $\|x + y\| \leq \|x\| + \|y\|$
  - $\|\alpha x\| = |\alpha| \|x\|$

- A linear space $N$, with a norm defined on it, is called a Normed linear space.

Definition : Banach Space

- A complete normed linear space is called a Banach space.

Results :

1. $\|x\| - \|y\| \leq \|x - y\|$
2. Norm is a continuous function on $N$
   
   i.e., $\|x_n - x\| \rightarrow 0$ as $n \rightarrow \infty$
3. In a normed linear space $N$, addition and scalar multiplication are jointly continuous.
   
   i.e., if $x_n \rightarrow x, y_n \rightarrow y$ and $\alpha_n \rightarrow \alpha$ then
   $x_n + y_n \rightarrow x + y$ and $\alpha_n x_n \rightarrow \alpha x$

Example for Banach Spaces

1. The real linear space $R$ and the complex linear space $C$ are Banach spaces under the norm defined by $\|x\| = |x|, \forall x \in R$ or $C$.
2. The linear spaces $R^n$ and $C^n$ are Banach spaces under the norm defined by
   $\|x\| = (\sum_{i=1}^{n} |x_i|^2)^{\frac{1}{2}}, \forall x$
   $R^n \rightarrow n$ –Dimensional Euclidean space
   $C^n \rightarrow Unitary$ space

3. The space of n-tuples of scalars $l^n_p$ is a Banach space with the norm
   \[ \|x\|_p = \left( \sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \]

4. The linear space $l_p$ of all sequence $x = \{x_1, x_2, \ldots, x_n, \ldots\}$ of scalars such that $\sum_{n=1}^{\infty} |x_n|^p < \infty$ is a Banach space under the norm defined by
   \[ \|x\|_p = \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} \]

5. The linear space $l^\infty_n$ is a Banach space under the norm
   \[ \|x\|_\infty = \max\{|x_1|, |x_2|, \ldots, |x_n|\} \]

6. The linear space $l^\infty$ of all bounded sequence $x = \{x_1, x_2, \ldots, x_n, \ldots\}$ of scalar is a Banach space under the norm
   \[ \|x\| = \sup_{n=1,2,\ldots} |x_n| \]

7. The linear space $C(x)$ of all bounded continuous scalar valued functions defined on a topological space is a Banach space under the norm
   \[ \|f\| = \sup_{x \in X} |f(x)|, f(x) \in C(x) \]

8. The space $L_p$ of all measurable functions defined on a measure space $X$ with the property that $|f(x)|^p$ is integrable is a Banach space under the norm
   \[ \|f\|_p = \left( \int |f(x)|^p d\mu(x) \right)^{\frac{1}{p}} \]

**Theorem**

Let $M$ be a closed subspace of a normed linear space $N$ and the norm of a coset $x + M$ in the quotient space $N/M$ is defined as
   \[ x + M = \inf\{\|x + m\|: m \in M\}. \]

Then $N/M$ is a normed linear space. If in addition, if $N$ is a Banach space, then so is $N/M$.

**Holder’s Inequality**:

\[ \sum_{i=1}^{n} |x_i y_i| \leq \left( \sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^{n} |y_i|^q \right)^{\frac{1}{q}} \]

i.e., $\sum_{i=1}^{n} |x_i y_i| \leq \|x\|_p \|y\|_q$

where $p > 1, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$

**Minkowski Inequality**

\[ \left( \sum_{i=1}^{n} |x_i y_i|^p \right)^{\frac{1}{p}} \leq \left( \sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^{n} |y_i|^p \right)^{\frac{1}{p}} \]

i.e., $\|x + y\|_p \leq \|x\|_p + \|y\|_p$

**Definition (Closed unit sphere)**

Let $X$ be a Banach space. A closer unit sphere in $X$ is defined as
   \[ S = \{ x \in X: \|x\| \leq 1 \} \]

**Result**: $S$ is a convex set.
Some example for closed sphere:
1. The linear space $\mathbb{R}^2$, the set of all ordered pairs, is a Banach space under the norm $\|x\| = |x_1| + |x_2|$. The closed unit sphere of $\mathbb{R}^2$ is $S = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$ i.e., $S = \{(x_1,x_2) \in \mathbb{R}^2 : |x_1| + |x_2| \leq 1\}$

2. In $\mathbb{R}^2$, the norm is defined by $\|x\| = (|x_1|^2 + |x_2|^2)^{\frac{1}{2}}$ We derive a closed unit sphere.

3. In $\mathbb{R}^2$, the norm is defined by $\|x\|_\infty = \max\{|x_1|,|x_2|\}$. We will derive a closed unit sphere.

4. We define the norm
$$\|x\|_p = (|x_1|^p + |x_2|^p)^{\frac{1}{p}}, 1 \leq p < \infty$$
then $S$ is truly spherical iff $p = 2$

In this case, $p < 1$, then

$S = \{x : \|x\| \leq 1\}$ would not be convex.

**Definition (Linear transformation)**

Let $N$ and $N'$ be two linear spaces over the same system of scalars. A mapping

$T : N \rightarrow N'$ is called a linear transformation if

i) $T(x + y) = T(x) + T(y)$

ii) $T(ax) = aT(x), \forall x, y \in N$

$\alpha$ a scalar

Equivalently,

$T(ax + \beta y) = aT(x) + \beta T(y), \forall x, y \in N$

and $\alpha, \beta$ scalars

**Note**: $T(0) = 0$

**Definition (Continuous linear transformation)**

Let $N$ and $N'$ be two normed linear spaces. Let $T : N \rightarrow N'$ be a linear transformation if whenever $\{x_n\}$ is a sequence in $N$ such that $x_n \rightarrow x$ in $N$, then the sequence $\{T(x_n)\}$ converges to $T(x)$ in $N'$

i.e., $x_n \rightarrow x \Rightarrow T(x_n) \rightarrow T(x)$

**Theorem**

Let $N$ and $N'$ be two normed linear spaces. Let $T : N \rightarrow N'$ be a linear transformation, then the following conditions on $T$ are equivalent to one another:

i) $T$ is continuous

ii) $T$ is continuous at the origin in the sense that $x_n \rightarrow 0 \Rightarrow T(x_n) \rightarrow 0$

iii) There exists a real number $K \geq 0$ such that $\|T(x)\| \leq K\|x\|, \forall x \in N$

iv) If $S = \{x : \|x\| \leq 1\}$ is the closed unit sphere in $N$, then its image $T(S)$ is a bounded set in $N'$

**Remarks**

i) By condition (iii) of the above theorem, there exists a real number $K \geq 0$ such that $\|T(x)\| \leq K\|x\|, \forall x \in N$

This $K$ is called a bound for $T$ and hence the linear transformation $T$ is bounded linear transformation.
ii) By the above theorem, $T$ is continuous if $T$ is bounded interchangeably

**Definition :** If $T$ is a continuous linear transformation of $N$ into $N'$, then norm of $T$ is defined as $\|T\| = \text{Sup}\{\|T(x)\| : \|x\| \leq 1\}$

**Remarks**

i) If $N \neq \{0\}$, we can give another equivalent expression of $\|T\|as \|T\| = \text{Sup}\{\|T(x)\| : \|x\| \geq 1\}$

ii) By the conditions (iii) and (iv) of the above theorem

We arrive at $\|T(x)\| \leq \|T\|\|x\|, \forall x \in N$

**Notation**

The set of all continuous (bounded) linear transformation from $N$ into $N'$ is denoted by $\mathcal{B}(N,N')$

**Theorem**

Let $N$ and $N'$ be normed linear space then $\mathcal{B}(N,N')$ is a normed linear space with respect to the pointwise linear operation

$$(T + v)(x) = T(x) + v(x)$$

$$(\alpha T)(x) = \alpha T(x)$$

and with the norm defined by

$\|T\| = \text{Sup}\{\|T(x)\| : \|x\| \leq 1\}$

further if $N'$ is a Banach space, then $\mathcal{B}(N,N')$ is also a Banach space.

**Definition : Algebra**

A linear space $N$ is called an algebra if its element can be multiply in such way that $N$ is a ring and the scalar multiplication is related with the multiplication by the way $a(xy) = (ax)y = x(ay)$

**Definition : operator**

Let $N$ be a normed linear space. A continuous linear transformation of $N$ into itself is called an operator on $N$. The set of all operators of a normed linear space $N$ is denoted by $\mathcal{B}(N)$

**Results :**

1. If $N$ is a Banach space, then $\mathcal{B}(N)$ is also a Banach space
2. $\mathcal{B}(N)$ is an algebra where the multiplication of its elements is defined as $(Tv)(x) = T(vx), \forall v, T \in \mathcal{B}(N)$ and $x \in N$
3. In $\mathcal{B}(N)$, multiplication is jointly continuous i.e.,

   $Tn \rightarrow T$ and $Tn' \rightarrow T' \Rightarrow TnTn' \rightarrow TT'$

4. If $N \neq \{0\}$, then the identity transformation $I$ of $N$ into itself is the identity element of the algebra

   $\mathcal{B}(N)$ and $\|I\| = 1$
Definition
Let $N$ and $N'$ be normed linear spaces
1. An isometric isomorphism of $N$ to $N'$ is a one-to-one linear transformation of $N$ into $N'$ such that $\|T(x)\| = \|x\|, \forall x \in N$
2. $N$ is isometrically isomorphic to $N'$ if there exists an isometric isomorphism of $N$ onto $N'$

Results
1. If $M$ is closed subspaces of a normed linear space $N$ and if $T$ is a natural mapping of $N$ onto $N/M$ defined by $T(x) = x + M, \forall x \in N$, then $T$ is a continuous linear transformation for which $\|T\| \leq 1$
2. If $T$ is a continuous linear transformation from a normed linear space to another normed linear space $N'$ and if $M$ is its null space, then there exists a natural linear transformation $T'$ of $N/M$ into $N'$ which is such that $\|T'\| = \|T\|

Theorem
Let $M$ be a linear subspace of a normed linear space $N$ and let $f$ be a functional defined on $M$. Then $f$ can be extended to functional $f_0$ defined on the whole spaces $N$ such that $\|f\| = \|f_0\|$ 

Conjugate space of $L_p$
Let $X$ be a measure space with measure $m$ and $p$ be a general real number such that $1 < p < \infty$ Consider the Banach space of $L_p$ of all measurable functions $f$ defined on $x$ such that $|f(x)|^p$ is integrable

Let $g$ be an element of $L_q$ where $\frac{1}{p} + \frac{1}{q} = 1$
Define a function $F_g$ on $L_p$ by

$$F_g(f) = \int f(x)g(x)dm(x)$$
$$\|F_g(f)\| = \left|\int f(x)g(x)dm(x)\right|$$
$$\leq \int |f(x)g(x)dm(x)|$$

$|fg(f)| \leq \|f\|_p \|g\|_q,$
(by Holder's inequality)

Taking Sup for all functions
$$f \in L_p \text{ such that } \|f\|_p \leq 1$$
We get $\|F_g\| \leq \|g\|_q$

- It shows that $F_g$ is well defined a scalar valued continuous linear function on $L_p$
- with the property that $\|F_g\| \leq \|g\|_q$

• It can also be shown that every linear functional $L_p$ arises in this way

• Hence $g \rightarrow L_p$ is linear and is an isometric isomorphism of $L_q$ into $L_p^*$

• This we write as $L_p^* = L_q$

Remarks

1. If we consider that such specialization to the space of all n-tuples of scalars we have $(l_p^n)^* = l_p^*$

   If $p = 1$, we have $(L_1^n)^* = L_\infty^n$

2. If we specialize to the space of all sequence of scalar, we have $l_p^* = l_q$

   When $p = 1, l_1^* = l_\infty$ and $C_0^* = l_1$

Theorem

If $M$ is a closed linear subspace of a normed linear space $N$ and $x_0$ is not a vector in $M$, then there exists a functional $f_0$ in $N^*$ such that $f(M) = 0$ and $f_0(x_0) \neq 0$

(i.e., Banach space has rich supply functionals)

Result

Let $M$ be a closed linear subspace of a normed linear space $N$ and let $x_0$ be a vector not in $M$, if $d$ is the distance from $x_0$ to $M$.

Then there exists a functional $f_0$ in $N^*$ such that $f_0(M) = 0$ and $f(x_0) = 1$ and $\|f_0\| = \frac{1}{d}$

Definition (second conjugate space, $N^{**}$)

Let $N$ be a normed linear space. Then the conjugate space of the conjugate space is defined by $N^{**}$ is called the second conjugate space of $N$

Result

Let $N$ be a normed linear space. Then each vector $x$ in $N$ induces a functional $F_x$ on $N^*$ defined by $F_x(f) = f(x)$ for all $f \in N^*$

Such that $\|F_x\| = \|x\|

Then the mapping $f: N \rightarrow N^{**}$
(i.e., $x \to F_x$) defines an isometric isomorphism of $N$ into $N^{**}$

**Remarks**

1. The functional $F_x$ in the above result, is called induced functional.
2. The isometric isomorphism $x \to F_x$ is called the normal imbedding of $N$ in $N^{**}$, for it allows as to record $N$ as a part of $N^{**}$ without altering any of its structure as a normed linear space.

Hence we write $N \subseteq N^{**}$

**Definition**

A normed linear space $N$ is said to be reflexive if $N \equiv N^{**}$ (i.e., the above isometric isomorphic is onto also)

The spaces $l_p$ for $1 < p < \infty$ are reflexive for $l_p^* = l_q$ but $l_q^* = l_p$

i.e., $(l_p^*)^* = l_p$

i.e., $l_p^{**} = l_p$

**Remarks**

1. Since $N^{**}$ is complete, $N$ is necessarily complete if it is reflexive.
2. But if $N$ is complete, then it need not be reflexive.

For example

$$C_0^* = l_1 \text{ and } C_0^{**} = l_1^* = l_\infty$$

**Theorem**

If $B$ and $B'$ are Banach space and if $T$ is continuous linear transformation of $B$ onto $B'$, then the image of each open sphere centred on the origin in $B$ contains an open sphere centred on the origin in $B'$

**Theorem (The open mapping theorem)**

If $B$ and $B'$ are Banach space if $T$ is a continuous linear transformation of $B$ onto $B'$, then $T$ is an open mapping

(i.e., $G$ is open in $B \implies T(G)$ is open in $B'$)

**Theorem**

A one-to-one continuous linear transformation of one Banach space onto another is a homomorphism.