TRB – PG (MATHS)
Complete Study Material

- Unit Wise - Notes
- Unit Wise Questions
- Previous Year Question Paper (Problem Solved)
- Model Questions
- Education (Jeba's Education Guide)
- Gk (Special Jeba's GK guide with only full material)

மாதா பதிப்பகம்

Unit – VI

Functional Analysis

Banach Spaces - Definition and example - continuous linear transformations - Banach theorem - Natural embedding of X in X - Open mapping and closed graph theorem - Properties of conjugate of an operator - Hilbert spaces - Orthonormal bases - Conjugate space H - Adjoint of an operator – Projections l^2 as a Hilbert space – l^p space - Holders and Minkowski inequalities - Matrices – Basic operations of matrices - Determinant of a matrix - Determinant and spectrum of an operator - Spectral theorem for operators on a finite dimensional Hilbert space - Regular and singular elements in a Banach Algebra – Topological divisor of zero - Spectrum of an element in a Banach algebra - the formula for the spectral radius radical and semi simplicity.

Banach Spaces

Definition (Normed Linear Space)

- Let N be a linear spaces. A norm on N is a real function $\|\|: N \to R$ satisfying the following conditions:

  For $x, y \in N$ and $\propto a\ scalar,$

- $\|x\| \geq 0\ and\ \|x\| = 0 \Leftrightarrow x = 0$
- $\|x + y\| \leq \|x\| + \|y\|$
- $\|\alpha x\| = |\alpha|\|x\|$
- A linear space N, with a norm defined on it, is called a Normed linear space.

Definition : Banach Space

- A complete normed linear space is called a Banach space.

Results :

1. $\|x\| - \|y\| \leq \|x - y\|$
2. Norm is a continuous function on N

   i.e., $\|x_n - x\| \to 0\ as\ n \to \infty$
3. In a normed linear space N, addition and scalar multiplication are jointly continuous.

   i.e., if $x_n \to x, y_n \to y\ and\ \alpha_n \to \alpha\ then$

   $x_n + y_n \to x + y\ and\ \alpha_n x_n \to \alpha x$

Example for Banach Spaces

1. The real linear space R and the complex linear space C are Banach spaces under the norm defined by $\|x\| = |x|, \forall x \epsilon R\ or\ C.$
2. The linear spaces $R^n\ and\ C^n$ are Banach spaces under the norm defined by

   $\|x\| = (\sum_{i=1}^{n} |x_i|^2)^{\frac{1}{2}},\ \forall x$

   $R^n \to n$ –Dimensional Euclidean space

   $C^n \to$ Unitary space

3. The space of n-tuples of scalars l_p^n is a Banach space with the norm

   $\|x\|_p = (\sum_{i=1}^{n}|x_i|^p)^{\frac{1}{p}}$

4. The linear space l_p of all sequence $x = \{x_1, x_2, \ldots\ldots x_n, \ldots.\}$ of scalars such that $\sum_{n=1}^{\infty}|x_n|^p < \infty$ is a Banach space under the norm defined by

   $\|x\|_p = (\sum_{n=1}^{\infty}|x_n|^p)^{\frac{1}{p}}$

5. The linear space $l_\infty{}^n$ is a Banach space under the norm

   $\|x\|_\infty = max\{\{|x_1|, |x_2|, \ldots\ldots |x_n|\}\}$

6. The linear space l_∞, of all bounded sequence $x = \{x_1, x_2, \ldots\ldots x_n, \ldots.\}$ of scalar is a Banach space under the norm

   $\|x\| = Sup_{n=1,2\ldots}|x_n|$

7. The linear space $C(x)$ of all bounded continuous scalar valued functions defined on a topological space is a Banach space under the norm

   $\|f\| = Sup_{x\epsilon X}|f(x)|, f(x)\epsilon C(x)$

8. The space L_P of all measurable functions defined on a measure space X with the property that $|f(x)|^p$ is integrable is a Banach space under the norm

$$\|f\|_p = \left(\int |f(x)|d_m(x)\right)^{\frac{1}{p}}$$

Theorem

Let M be a closed subspace of a normed linear space N and the norm of a coset $x + M$ in the quotient space N/M is defined as $x + M = inf\{\|x + m\|: m\,\epsilon\, M\}$. Then N/m is a normed linear space. If in addition, if N is a Banach space, then so is N/M.

Holder's Inequality :

$$\boxed{\sum_{i=1}^{n}|x_i y_i| \leq \left(\sum_{i=1}^{n}|x_i|^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}|y_i|^q\right)^{\frac{1}{q}}}$$

i.e., $\sum_{i=1}^{n}|x_i y_i| \leq \|x\|_p\|y\|_q$

where $p > 1, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$

Minkowski Inequality

$$\boxed{\left(\sum_{i=1}^{n}|x_i y_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n}|x_i|^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}|y_i|^p\right)^{\frac{1}{p}}}$$

i.e., $\|x + y\|_p \leq \|x\|_p + \|y\|_p$

Definition (Closed unit sphere)

Let X be a Banach space. A closer unit sphere in X is defined as

$$S = \{x\epsilon X: \|x\| \leq 1\}$$

Result : S is a convex set.

Some example for closed sphere:

1. The linear space R^2, the set of all ordered pairs, is a Banach space under the norm $\|x\| = |x_1| + |x_2|$. The closed unit sphere of R^2 is $S = \{x \epsilon R^2 : \|x\| \leq 1\}$ i.e., $S = \{(x_1, x_2) \epsilon R^2 : |x_1| + |x_2| \leq 1\}$
2. In R^2, the norm is defined by $\|x\|_2 = (|x_1|^2 + |x_2|^2)^{\frac{1}{2}}$ We derive a closed unit sphere.
3. In R^2, the norm is defined by $\|x\|_\infty = \max\{|x_1|, |x_2|\}$. We will derive a closed unit sphere.
4. We define the norm $\|x\|_p = (|x_1|^p + |x_2|^p)^{\frac{1}{p}}, 1 \leq p < \infty$ then S is truly spherical iff $p = 2$
   In this case, $p < 1$, then
   $S = \{x : \|x\| \leq 1\}$ would not be convex.

Definition (Linear transformation)

Let N and N' be two linear spaces over the same system of scalars. A mapping T: N → N' is called a linear transformation if

i) $T(x + y) = T(x) + T(y)$

ii) $T(\alpha x) = \alpha T(x), \forall x, y \; \epsilon N$

$\alpha \; a \; scalar$

Equivalently,

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y), \forall x, y \; \epsilon N$$
$$and \; \alpha, \beta \; scalars$$

Note : $T(0) = 0$

Definition (Continuous linear transformation)

Let N and N' be two normed linear spaces. Let T: N → N' be a linear transformation if whenever $\{x_n\}$ is a sequence in N such that $x_n \to x$ in N, then the sequence $\{T(x_n)\}$ converges to $T(x) in \; N'$

i.e., $x_n \to x \Longrightarrow T(x_n) \longrightarrow T(x)$

Theorem

Let N and N' be two normed linear spaces. Let T: N → N' be a linear transformation, then the following conditions on T are equivalent to one another:

i) T is continuous

ii) T is continuous at the origin in the sense that $x_n \longrightarrow 0 \Longrightarrow T(x_n) \longrightarrow 0$

iii) There exists a real number $K \geq 0$ such that $\|T(x)\| \leq K\|x\|, \forall x \epsilon N$

iv) If $S = \{x : \|x\| \leq 1\}$ is the closed unit sphere in N, then its image T(S) is a bounded set in N'

Remarks

i) By condition (iii) of the above theorem, there exists a real number $K \geq 0$ such that
$\|T(x)\| \leq K\|x\|, \forall x \epsilon N$
This K is called a bound for T and hence the linear transformation T is bounded linear transformation

ii) By the above theorem, T is continuous if T is bounded interchangeably

Definition : If T is a continuous linear transformation of N into N', then norm of T is defined as $\|T\| = Sup\{\|T(x)\|: \|x\| \leq 1\}$

Remarks

i) If $N \neq \{0\}$, we can give another equivalent expression of $\|T\| as\ \|T\| = Sup\{\|T(x)\|: \|x\| \geq 1\}$

ii) By the conditions (iii) and (iv) of the above theorem

We arrive at

$$\|T(x)\| \leq \|T\|\|x\|, \forall x \epsilon N$$

Notation

The set of all continuous (bounded) linear transformation from N into N' is denoted by $\mathcal{B}(N, N')$

Theorem

Let N and N' be normed linear space then $\mathcal{B}(N, N')$ is a normed linear space with respect to the pointwise linear operation

$$(T + v)(x) = T(x) + v(x)$$

$$(\alpha T)(x) = \alpha T(x)$$

and with the norm defined by

$$\|T\| = Sup\{\|T(x)\|: \|x\| \leq 1\}$$

further if N' is a Banach space, then $\mathcal{B}(N, N')$ is also a Banach space.

Definition : Algebra

A linear space N is called an algebra if its element can be multiply in such way that N is a ring and the scalar multiplication is related with the multiplication by the way $\alpha(xy) = (\alpha x)y = x(\alpha y)$

Definition : operator

Let N be a normed linear space. A continuous linear transformation of N into itself is called an operator on N. The set of all operators of a normed linear space N is denoted by $\mathcal{B}(N)$

Results :

1. If N is a Banach space, then $\mathcal{B}(N)$ is also a Banach space
2. $\mathcal{B}(N)$ is an algebra where the multiplication of its elements is defined as $(Tv)(x) = T(vx), \forall v, T \epsilon \mathcal{B}(N)$ and $x \epsilon N$
3. In $\mathcal{B}(N)$, multiplication is jointly continuous i.e.,
$$Tn \to T \text{ and } Tn' \to T' \Longrightarrow TnTn' \longrightarrow TT'$$
4. If $N \neq \{0\}$, then the identity transformation I of N into itself is the identity element of the algebra
$$\mathcal{B}(N) and\ \|I\| = 1$$

Definition

Let N and N' be normed linear spaces

1. An isometric isomorphism of N to N' is a one-to-one linear transformation of N into N' such that $\|T(x)\| = \|x\|, \forall x \epsilon R$
2. N is isometrically isomorphic to N' if T an isometric isomorphism of N onto N'

Results

1. If M is closed subspaces of a normed linear space N and if T is a natural mapping of N onto N/M defined by $T(x) = x + M, \forall x \epsilon N$, then T is a continuous linear transformation for which $\|T\| \leq 1$
2. If T is a continuous linear transformation from a normed linear space to another normed linear space N' and if M is its null space, then there exists a natural linear transformation T' of N/M into N' which is such that $\|T'\| = \|T\|$

Theorem

Let M be a linear subspace of a normed linear space N and let f be functional defined on M. If x_0 is not a vector in M and if $M_0 = M + x_0$ is the linear subspace generated By $M \text{ and } x_0$, then f can be extended to a functional $f_0 \text{ on } M_0$

$$such\ that\ \|f\| = \|f_0\|$$

Theorem (The Hahn-Banach theorem)

Let M be a linear subspaces of a normed linear space N and let f be a functional defined on M. Then f can be extended to functional f_0 defined on the whole spaces N such that $\|f\| = \|f_0\|$

Conjucate space of L_p

Let X be a measure space with measure m and p be a general real number such that $1 < p < \infty$ Consider the Banach space of l_p of all measurable functions f defined on x such that $|f(x)|^p$ is integrable

Let g be an element of L_q where $\frac{1}{p} + \frac{1}{q} = 1$

Define a function F_g on L_p by

$$F_g(f) = \int f(x)g(x)d_m(x)$$

$$|F_g(f)| = \left|\int f(x)g(x)d_m(x)\right|$$

$$\leq \int |f(x)g(x)d_m(x)|$$

$$|fg(f)| \leq \|f\|_p \|g\|_q,$$

$$(by\ Holder's\ inequality)$$

Taking Sup for all functions

$f \epsilon Lp\ such\ that \|f\|_p \leq 1$

We get $\|F_g\| \leq \|g\|_q$

- It shows that F_g is well defined a scalar valued continuous linear function on L_p with the property that $\|F_g\| \leq \|g\|_q$

- It can be also be shown that every linear functional L_p arises in this way
- Hence $g \rightarrow L_p$ is linear and is an isometric isomorphism of L_q into L_p*
- This we write as $L_p^* = L_q$

Remarks

1. If we consider that such specialization to the space of all n-tuples of scalars we have $\left(l_p{}^n\right)^* = l_p{}^*$
   $If\ p = 1, we\ have\ (L_1{}^n)^* = L_\infty^n$
2. If we specialize to the space of all sequence of scalar, we have $l_p{}^* = l_q$
   When $p = 1, l_1{}^* = l_\infty\ and\ C_0^* = l_1$

Theorem

If N is a normed linear space and if x_0 is a non-zero vector in N, then there exist a functional f_0 in N* such that

$f(x_0) = \|x_0\|\ and\ \|f_0\| = 1$

Remarks

1. N* seperates vectors in N
   i.e., let $x, y \epsilon N\ with\ x \neq y$
   then there exists a function in N^* such that $f(x) \neq f(y)$

Theorem

If M is a closed linear subspace of a normed linear space N and x_0 is not a vector in M, then there exists a functional $f_0\ in\ N^*$

$such\ that\ f(M) = 0\ and\ f_0(x_0) \neq 0$

(i.e, Banach space has rich supply functionals)

Result

Let M be a closed linear subspace of a normed linear space N and let x_0 be a vector not in M. if d is the distance from $x_0\ to\ M$. Then there exists a functional $f_0\ in\ N^*\ such\ that$

$$f_0(M) = 0\ and\ f(x_0) = 1\ and\ \|f_0\| = \frac{1}{d}$$

Definition (second conjugate space, N)**

Let N be a normed linear space. Then the conjugate space of the conjugate space is defined by N** is called the second conjugate space of N

Result

Let N be a normed linear space. Then each vector x in N induces a functional F_x on N* defined by $F_x(f) = f(x)\ for\ all\ f \epsilon N^*$

Such that $\|f_x\| = \|x\|$

Then the mapping $J: N \rightarrow N^{**}$

($i.e., x \to F_x$) defines an isometric isomorphism of N into N**

Remarks

1. The functional F_x, in the above result, is called induced functional.
2. The isometric isomorphism $x \to F_x$ is called the normal imbedding of N in N**, for it allows as to record N as a part of N** without altering any of its structure as a normed linear space. Hence we write $N \subseteq N^{**}$

Definition

A normed linear space N is said to be reflexive if $N \equiv N^{**}$
(i.e., the above isometric isomorphic is onto also)

The spaces $l_p\ for\ 1 < p < \infty$ are reflexive for $l_p{}^* = l_q\ but\ l_q{}^* = l_p$

i.e., $\left(l_p{}^*\right)^* = l_p$

i.e., $l_p{}^{**} = l_p$

Remarks

1. Since N** is complete, N is necessarily complete if it is reflexive.
2. But if N is complete, then it need not be reflexive.

   For example

$$C_0{}^* = l_1\ and\ C_0{}^{**} = l_1{}^* = l_\infty$$

Theorem

If B and B' are Banach space and if T is continuous linear transformation of B onto B', then the image of each open sphere centred on the origin in B contains an open sphere centred on the origin in B'

Theorem (The open mapping theorem)

If B and B' are Banach space if T is a continuous linear transformation of B onto B', then T is an open mapping

$(i.e., G\ is\ open\ in\ B \Longrightarrow T(G) is\ open\ in\ B')$

Theorem

A one-to-one continuous linear transformation of one Banach space onto another is a homomorphism.