

NATIONAL ACADEMY

DHARMAPURI

TRB MATHEMATICS

FUNTIONAL ANALYSIS

‘Material Available with Question papers’

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TEST BACTH –SATURDAY & SUNDAY

NATIONAL ACADEMY

DHARMAPURI

CLASS -I**Holder's inequality**

If $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then $\sum_{i=1}^n |x_i y_i| \leq [\sum_{i=1}^n |x_i|^p]^{\frac{1}{p}} [\sum_{i=1}^n |y_i|^q]^{\frac{1}{q}}$ OR

$$\sum_{i=1}^n |x_i y_i| \leq [\sum_{i=1}^n |x_i|] [\sum_{i=1}^n |y_i|]$$

Holder's inequality For integrable function

$$\int_a^b |f(x)g(x)| dx \leq [\int_a^b |f(x)|^p dx]^{\frac{1}{p}} [\int_a^b |g(x)|^q dx]^{\frac{1}{q}}$$

put $p = q = 2$, then,

$$\sum_{i=1}^n |x_i y_i| \leq [\sum_{i=1}^n |x_i|^2]^{\frac{1}{2}} [\sum_{i=1}^n |y_i|^2]^{\frac{1}{2}} \text{ or}$$

$$[\sum_{i=1}^n |x_i y_i|]^2 \leq [\sum_{i=1}^n |x_i|^2] [\sum_{i=1}^n |y_i|^2]$$

This is known as Cauchy's inequality

Minkowsk's inequality

➤ If $p \geq 1$, then $[\sum_{i=1}^n |x_i + y_i|^p]^{\frac{1}{p}} \leq [\sum_{i=1}^n |x_i|^p]^{\frac{1}{p}} + [\sum_{i=1}^n |y_i|^p]^{\frac{1}{p}}$

➤ If f and g are real or complex valued integrable function defined on $[a, b]$, Then

$$[\int_a^b |f(x) + g(x)| dx]^p \leq [\int_a^b |f(x)|^p dx]^{\frac{1}{p}} + [\int_a^b |g(x)|^q dx]^{\frac{1}{q}} \text{ where } p \geq 1$$

Metric space

Let X be a non-empty set. A metric on X is a real valued function $X \times X$ satisfying the following Three conditions,

For every $x, y \in X$ and $x \neq y$

1. $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$
2. $d(x, y) = d(y, x)$ for every $x, y \in X$
3. $d(x, y) \leq d(x, z) + d(z, y)$ for any $x, y, z \in X$

$d(x, y)$ is called the distance between x and y , it is finite non-negative real number.

Normed linear spaces

Let N be a complex or real linear space a norm on N is a function such that

$$(\| \cdot \| : N \rightarrow R)$$

- i. $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$

- ii. $\|x + y\| \leq \|x\| + \|y\|$
 iii. $\|ax\| = |a|\|x\|$ for all $x, y \in N$ and $a \in C$ or R

N is called a normed linear space.

Definition

Let N be a normed linear space, a sequence $\{x_n\}$ in N is said to converge to an element x in N if given $\varepsilon > 0$, there exists a positive integer n_0 such that

$$\|x_n - x\| < \varepsilon \text{ for all } n \geq n_0$$

It is denoted by $\lim_{n \rightarrow \infty} x_n = x$

$x_n \rightarrow x$ iff $\|x_n - x\| \rightarrow 0$ as $n \rightarrow \infty$

Theorems

- A normed linear space N is a metric space with respect to the metric d defined by

$$D(x, y) = \|x - y\| \text{ for all } x, y \in N$$

- If N is a normed linear space, Then

- $\| \|x\| + \|y\| \| \leq \|x\| + \|y\|$

- $\| \|x\| - \|y\| \| \leq \|x - y\|$

- If N is a normed linear space, Then the norm $\| \cdot \|: N \rightarrow R$ is continuous on N .

- The operation of addition and scalar multiplication in N are jointly continuous.

If $x_n \rightarrow x$, $y_n \rightarrow y$ and $a_n \rightarrow a$, Then $x_n + y_n \rightarrow x + y$, $a_n x_n \rightarrow ax$

- Let N be a normed linear space and M be a subspace of N , then the closure \bar{M} of M is also a subspace of N

- A subset M in a normed linear space N is bounded if and only if there is a positive constant C such that $\|x\| \leq C$ for all $x \in M$

Cauchy sequence

A sequence $\{x_n\}$ in N is called a Cauchy sequence in N , If given $\varepsilon \geq 0$ there exists a positive integer n_0 such that $\|x_n - x_m\| < \varepsilon$ for all $m, n \geq n_0$

If $\{x_n\}$ is a Cauchy sequence in N , Then $\|x_n - x_m\| \rightarrow 0$ as $m, n \rightarrow \infty$

Properties of a Cauchy sequence

i. If N is normed linear space, then every convergent sequence is a Cauchy sequence.

It's converse is not true

ii. Every Cauchy sequence in a normed linear space is bounded.

Complete

A normed linear space N is said to be complete if every Cauchy sequence in N converges to an element of N .

\Rightarrow If $\|x_n - x_m\| \rightarrow 0$ as $n, m \rightarrow \infty$, then there exists $x \in N$ such that

$$\|x_n - x\| \rightarrow 0 \text{ as } n \rightarrow \infty,$$

Banach space

A complete normed linear space is called a Banach space.

➤ Every complete subspace M of a normed linear space is closed

Convergent of series

A series $\sum_{n=1}^{\infty} x_n$, $x_n \in N$ is said to be **convergent to $x \in N$** , if the sequence of partial sums $\{s_n\}$ converges to x in N .

A series $\sum_{n=1}^{\infty} x_n$ is said to be **absolutely convergent** if $\sum_{n=1}^{\infty} \|x_n\|$ is convergent.

Theorem

A normed linear space N is complete if and only if every absolutely convergent series is convergent.

Example of Banach spaces.

1. The real linear space R and the complex linear space C are normed linear space under the norm $\|x\| = |x|$ for all $x \in R$ (or) C

R and C are complete $\Rightarrow R$ and C are Banach spaces.

2. The linear space R^n or C^n are Banach space with Norm, $\|x\| = [\sum_{i=1}^n |x_i|^2]^{\frac{1}{2}}$

3. (i) R^n or C^n are Banach space with Norm, $\|x\| = [\sum_{i=1}^n |x_i|^p]^{\frac{1}{p}}$, $1 \leq p \leq \infty$

Which is denoted by l_p^n

(ii) $\|x\| = \text{Max} \{ |x_1|, |x_2|, |x_3|, \dots, |x_n| \}$, which is denoted by l_∞^n

4. The linear space C of all convergent sequence $x = \{x_n\}$ with the Norm.

$$\|x\| = \sup_{1 \leq n \leq \infty} |x_n| \text{ is a Banach space denoted by } C$$

5. The linear space l_∞ of all bounded sequence $x = \{x_n\}$ with the Norm.

$$\|x\| = \sup_{1 \leq n \leq \infty} |x_n| \text{ is a Banach space.}$$

6. The linear space $l_p, p > 1$ of all sequences $[\sum_{i=1}^{\infty} |x_i|^p] < \infty$ with norm

$$\|x\| = [\sum_{i=1}^{\infty} |x_i|^p]^{\frac{1}{p}} \text{ is a Banach space, It is denoted by } \| \cdot \|_p$$

7. If $[a,b]$ is a bounded and closed interval, The linear space $C[a,b]$ of all continuous functions defined on $[a,b]$ is a Banach space with the norm,

$$\|f\| = \text{Sup}\{|f(x)| / x \in [a,b]\}$$

8. Let $C(X)$ be the set of all continuous real valued function on a compact metric space X , then $C(X)$ is a Banach space with the norm

$$\|f\| = \text{Sup}\{|f(x)| / x \in X\}$$

Separable

A normed linear space N is said to be separable if it has a countable dense subset.

ie., There is a countable subset D in N such that $\bar{D} = N$

Example

1. Every subset of a separable normal linear space is separable
2. The normed linear space $l_p, 1 \leq p \leq \infty$ are separable
3. The space l_∞ is not separable

Quotient space

Let N be a normed linear space and M be a subspace of N , Then $\frac{N}{M} = \{x + M / x \in N\}$ is called Quotient space.

It is denoted by $Q(x)$

$Q(x)$ is called canonical (Natural) mapping of L onto $\frac{N}{M}$

Theorem:

If M is a closed linear subspace of a normed linear space N , Then quotient space $\frac{N}{M}$ is a normed linear space with norm of each coset $x+M$ defined as

$$\|x + M\| = \inf\{ \|x + m\| / m \in M \}.$$

If N is Banach space, then the quotient space $\frac{N}{M}$ is also a Banach space with above norm

Direct sum of subspace

Let M and N are subspace of Banach space B , If every element z on B is represented uniquely in the form $z = x+y, x \in M, y \in N$, Then B is said to be direct sum of N, M

It is denoted by $B = M \oplus N$

Theorem

Let a Banach Space $B = M \oplus N$ and $z \in B$ be $z = x+y$ uniquely with $x \in M, y \in N$, then

$\|z\|_1 = \|x\| + \|y\|$ is a normal on direct sum $B = M \oplus N$

If B_1 is the direct sum space with this new norm, then B_1 is a Banach space if M and N are closed.

Continuous linear Transformation

$T: N \rightarrow N^1$ is continuous if and only if $x_n \rightarrow x$ in N implies $T(x_n) \rightarrow T(x)$ in N^1

1. Zero Transformation is denoted by 0
2. Identity Transformation is denoted by I

Theorem

If T is continuous at the origin, Then it is continuous everywhere and the continuity is uniform.

Bounded linear transformation

A linear transformation $T: N \rightarrow N^1$ is said to be bounded linear transformation if there exists a positive constant M such that $\|T(x)\| \leq M\|x\|$ for all $x \in N$.

Theorem

1. $T: N \rightarrow N^1$ is bounded if and only if T is continuous.
2. Let $T: N \rightarrow N^1$ be a linear transformation, Then T is bounded if and only if T maps bounded sets in N into bounded set in N^1

Bound of T

Let T be a bounded linear transformation of N into N^1 , Then the norm ,

$\|T(x)\| = \inf\{M / \|T(x)\| \leq M\|x\| \text{ for all } x \in N\}$ is called the bound of T (OR)

$$\|T\| = \sup\left\{\frac{\|T(x)\|}{\|x\|} / x \in N \text{ and } x \neq 0\right\}$$

Theorem

If N and N^1 are normed linear space and $T:N \rightarrow N^1$, Then the following are equivalent

- (a) $\|T\| = \sup\left\{\frac{\|T(x)\|}{\|x\|} / x \in N \text{ and } x \neq 0\right\}$
- (b) $\|T\| = \sup\{\|T(x)\| / x \in N \text{ and } \|x\| = 1\}$
- (c) $\|T\| = \sup\{\|T(x)\| / x \in N \text{ and } \|T(x)\| = 1\}$

$B(N, N^1)$

The set of all bounded linear transformation of normed space N into N^1 is denoted by $B(N, N^1)$

Theorem

➤ $B(N, N^1)$ is a normed linear space with linear operation

(i) $(T_1 + T_2)(x) = T_1(x) + T_2(x),$

(ii) $(aT)x = aT(x)$ and norm defined by $\|T\| = \sup\left\{\frac{\|T(x)\|}{\|x\|} / x \in N \text{ and } x \neq 0\right\}$

If N^1 is a Banach space, then $B(N, N^1)$ is also Banach space.

1. If $T_1, T_2 \in B(N, N^1)$, the

$$\|T_1 T_2\| \leq \|T_1\| \|T_2\|$$

2. If $T_n \rightarrow T$ and $T_n^{-1} \rightarrow T^{-1}$, Then $T_n T_n^{-1} \rightarrow T T^{-1}$ as $n \rightarrow \infty$ which implies that the multiplication is jointly continuous.

Theorem

➤ Let M be a closed subspace of a normed linear space and T be the natural mapping of N onto the quotient space $\frac{N}{M}$ defined by $T(x) = x + M$, Then T is bounded linear transformation with $\|T\| \leq 1$

➤ Let N and N^1 be normed linear space and let $T:N \rightarrow N^1$ be a bounded linear transformation of N into N^1 , If M is the kernel of T, then

i) M is closed subspace of N

ii) T induces a natural transformation T^1 of N/M onto N^1 such that $\|T^1\| = \|T\|$

Definition

Let N and N^1 be normed linear space, an **isometric isomorphism** of N into N^1 is a one-one linear transformation T of N into N^1 such that $\|T(x)\| = \|x\|$ for all $x \in N$

$$\text{For any } x, y \in N \Rightarrow \|T(x) - T(y)\| = \|T(x - y)\| = \|x - y\|$$

Definition**Topologically isomorphic**

Two normed linear space N and N^1 are said to be topologically isomorphic, if

- (i) There exists a linear operator $T: N \rightarrow N^1$ having the inverse T^{-1}
- (ii) T establishes the isomorphism of N and N^1
- (iii) T and T^{-1} are continuous in their respective domains.

Theorem

Let N and N^1 be normed linear space and Let T be linear transformation of N into N^1 . If $T(N)$ is the range of T , Then the inverse T^{-1} exists and is bounded (continuous) in its domain of definition if and only if there exists a constant $m > 0$ such that $m\|x\| \leq \|T(x)\|$ for all $x \in N$

Theorem

Let N and N^1 be normed linear space. The N and N^1 are topologically isomorphic if and only if there exist a linear operator T on N onto N^1 and positive constants m and M such that

$$m\|x\| \leq \|T(x)\| \leq M\|x\| \text{ for all } x \in N$$

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TRB MATHS COACHING CENTRE – DHARMAPURI

FUNCTIONAL ANALYSIS TEST – 1

1. If f and g are real or complex valued integrable function defined on $[a, b]$, Then Minkowsk's inequality is

$$(a) \left[\int_a^b |f(x) + g(x)| dx \right]^p \leq \left[\int_a^b |f(x)|^p dx \right]^{\frac{1}{p}} + \left[\int_a^b |g(x)|^q dx \right]^{\frac{1}{q}} \text{ where } p < 1$$

$$(b) \left[\int_a^b |f(x) + g(x)| dx \right]^p \leq \left[\int_a^b |f(x)| dx \right]^p + \left[\int_a^b |g(x)| dx \right]^p \text{ where } p \geq 1$$

$$(c) \left[\int_a^b |f(x) + g(x)| dx \right]^p \leq \left[\int_a^b |f(x)|^p dx \right]^{\frac{1}{p}} + \left[\int_a^b |g(x)|^p dx \right]^{\frac{1}{p}} \text{ where } p \geq 1$$

$$(d) \left[\int_a^b |f(x) + g(x)| dx \right]^p > \left[\int_a^b |f(x)|^p dx \right]^{\frac{1}{p}} + \left[\int_a^b |g(x)|^q dx \right]^{\frac{1}{q}} \text{ where } p \geq 1$$

2. If N be a complex or real linear space a norm on N is a function, Then

- (a) $\|x + y\| \leq \|x\| + \|y\|$ (b) $\|x + y\| > \|x\| + \|y\|$
 (c) $\|x + y\| + \|y\|$ (d) $\|x + y\| \leq \|x\|$
3. Let N be a normed linear space, For every $x, y \in N$
 (a) $|\|x\| - \|y\|| \leq \|x - y\|$ (b) $|\|x\| - \|y\|| > \|x\| - \|y\|$
 (c) $\|x - y\| = |\|x\| - \|y\||$ (d) $|\|x\| - \|y\|| = 0$
4. If every Cauchy sequence in N converges to an element of a normed linear space N , then N is
 (a) Banach space (b) complete (c) Hilbert space (d) Metric space
5. l_p^n is
 (a) Not Banach space (b) Linear space (c) Banach space (d) None of these
6. In a Banach space $x_n \rightarrow x$, $y_n \rightarrow y$ implies that $x_n + y_n \rightarrow$
 (a) $x + y$ (b) $\frac{x}{y}$ (c) $x - y$ (d) xy
7. If N be a Normed linear space and $\|x\| = 0$ if and only if
 (a) $x = 0$ (b) x is a real (c) $x \neq 0$ (d) $x > 0$
8. Every Cauchy sequence in a normed linear space is
 (a) not converges (b) absolutely convergent
 (c) bounded. (d) neither convergent nor divergent
9. A normed linear space N is complete if and only if every absolutely convergent series is,
 (a) not converges (b) convergent
 (c) divergent (d) neither convergent nor divergent
10. A subspace M of a Banach space B is complete if and only if M is
 (a) bounded (b) Unbounded (c) Closed in B (d) Open in B
11. If M is a closed linear subspace of a normed linear space N , Then quotient space $\frac{N}{M}$ is a normed linear space with norm
 (a) $\|x + M\| = \sup\{\|x + m\| / m \in M\}$ (b) $\|x + M\| = \inf\{\|x\| / x \in N\}$.
 (c) $\|x + M\| = \inf\{\|x + m\| / m \in M\}$ (d) $\|x + M\| = \inf\{\|m\| / m \in M\}$.
12. Let M be a closed subspace of a normed linear space N , For each $x \in N$,
 let $\|x + M\| = \inf\{\|x + m\| / m \in M\}$ then respective to this norm
 (a) $N + M$ is a normed linear space (b) NM is a normed linear space
 (c) $\frac{N}{M}$ is a normed linear space (d) $N - M$ is a normed linear space
13. A complete normed linear space is
 (a) Hilbert space (b) Banach space (c) Vector space (d) None of these

14. M is a closed linear subspace of the a normed linear space N . If N is a Banach space then the following is also a Banach space.
- (a) NM (b) $N+M$ (c) $N-M$ (d) $\frac{N}{M}$
15. If $p > 1$ and q is defined by $\frac{1}{p} + \frac{1}{q} = 1$ and for f and g two complex valued measurable function such that $f \in L_p(x), g \in L_q(x)$, then the Holder's inequality is
- (a) $\int_x |fg| dx \leq \|f\|_p \|g\|_q$ (b) $\left| \int_x fg dx \right| \leq \|f\|_p \|g\|_q$
(c) $\int_x |fg| dx \geq \|f\|_p \|g\|_q$ (d) $\left| \int_x fg dx \right| \geq \|f\|_p \|g\|_q$
16. Let M be a subspace of a normed linear space N . The set of all cosets $\{x+M/ x \in N\}$ is a normed space in the quotient form if
- (a) M is an open subspace of N (b) $M = N$
(c) M is a closed subspace of N (d) M is finite subspace os N
17. Let $(x_1, x_2, x_3, \dots, x_n) \in R^n$. $\|x\| = (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$ does not define a norm when
- (a) $p = 100$ (b) $p = \frac{3}{2}$ (c) $p = 1$ (d) $p = \frac{1}{2}$
18. Holder's inequality $\sum_{i=1}^n |x_i y_i| \leq [\sum_{i=1}^n |x_i|^p]^{\frac{1}{p}} [\sum_{i=1}^n |y_i|^q]^{\frac{1}{q}}$ for p, q such that ,
- (a) $p > 1$ and $p + q = 1$ (b) $p > 1$ and $\frac{1}{p} - \frac{1}{q} = 1$
(c) $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 0$ (d) $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$
19. If $1 \leq p_1 < p_2 < \infty$, then
- (a) $l_{p_1} \subset l_{p_2}$ and $\|x\|_{p_2} \geq \|x\|_{p_1}$ (b) $l_{p_1} \supset l_{p_2}$ and $\|x\|_{p_2} \leq \|x\|_{p_1}$
(c) $l_{p_1} \subset l_{p_2}$ and $\|x\|_{p_2} \leq \|x\|_{p_1}$ (d) $l_{p_1} \supset l_{p_2}$ and $\|x\|_{p_2} \geq \|x\|_{p_1}$
20. The linear space l_∞ of all bounded sequence $x = \{x_n\}$ is Banach space with the Norm.
- (a) $\|x\| = \text{Max} \{ |x_1|, |x_2|, |x_3|, \dots, |x_n| \}$ (b) $\|x\| = \sup_{1 \leq n < \infty} |x_n|$
(c) $\|x\| = [\sum_{i=1}^n |x_i|^p]^{\frac{1}{p}}$ (d) None of these
