# NATIONAL ACADEMY

# DHARMAPURI

## TRB MATHEMATICS

# ALGEBRA

**'Material Available with Question papers'**

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## CLASS -I

### Equivalence relation

The binary relation ~ on A is said to be an Equivalence relation on A if for all a,b,c in A

**1.Reflex**

$a \sim a$

**2. Symmstry**

$a \sim b \Rightarrow b \sim a$

**3. Transity**

$a \sim b \text{ and } b \sim c \Rightarrow a \sim c$

**Examples:**

1. Define $a \sim b$ for all a,b $\in S$ such that a = b , Than ~ is Equivalence relation on S
2. Define $a \sim b$ for all a,b $\in S$ such that a – b is even integer , Than ~ is Equivalence relation on S

### Equivalence class

The Equivalence class of a$\in A$ is the set $\{ x \in A \backslash a \sim x \}$

It is denoted by cl(a)

### Congruent modulo

Let n be a fixed positive integer. If a and b are integers such that a-b is divisible by n,We say that a is congruent to b modulo n and write a≡b(mod n)

Residue class modulo

$[a] = \{x \epsilon Z / x \equiv a(\text{mod } n) \}$

### Mapping

If A and B are nonempty sets, than a mapping from A to B is a subset of A$\times B$ such that for every a$\in A$ there is aunique b $\in B$ such that (a,b) $\in$ A$\times B$

map f:A$\rightarrow B$ , f(a) = b, where b is unique in B

### Onto mapping

The mapping f:A$\rightarrow B$ is said to be onto,if given b$\in B$ there exists an element a $\in A$ such that

f(a) = b

**One-to-one mapping**

The mapping f:A→ $B$ is said to be One-to-one mapping ,if whenever a=b,than f(a)=f(b) or a≠b,than f(a) ≠f(b)

**Composition(Product) of fuctions**

If f:A→ $B$ *and* g:B→ $C$, than Composition of f and g is a map g∘f : A→ $C$ defined by (g∘f)a = g[f(a)]

**Greatest common division (GCD)**

The positive integer c is said to be Greatest common division of a and b if

(i). c is a division of a and b (c\a and c\b )

(ii). Any division of a and b is a divisor of c (d\a and d\b ⇒ d\c )

it is denoted by (a,b) = c

**Relative prime**

The integers a ,b are called relatively prime, if (a,b) = 1

- If a and b are non zero integers, than (a,b) exists and we can find integers m ,n such that (a,b) = ma+nb
- If a,b are relatively prime,than there exists m,n such that ma+nb = 1 ( (a,b) = ma+nb=1 )

**Prime number**

The integer p> 1 is a prime number If its only divisors are $\pm 1, \pm p$

- If a is relatively prime to b and a\bc, then a\c

**Unique factorization**

Any positive integer a> 1 can be factored in a unique way as a $=p_1^{x} p_2^{y} \ldots \; p_n^{z}$ are prime numbers and each x> 0

**Division Algorithm**

Let a and b be integers, with b > 0. Then there exist unique integers q and r such that

a = bq + r where 0 ≤ r < b.

## GROUP:

A non-empty set G with abinary operation * is called agroup,If the following conditions are satisfied,

**1.Closure:** For all a,b $\epsilon$G ⇒a*b $\epsilon$G

**2.Associative**: For all a,b , $c$ $\epsilon$G ⇒a*(b*c) =(a*b)*c

**3.Identity:** For all a $\epsilon$G there exists an element e $\epsilon$G such that a*e =e*a =a

**4.Inverse:** For every a $\epsilon$G there exists an element $a^{-1}$ $\epsilon$ G such that a*$a^{-1}$ =$a^{-1}$*a =e

**Abelian group (or) commutative group:**

A group with commutative property is called an abelian group

That is, For all a,b $\epsilon$G ⇒a*b = b*a

**Semi group** : A set satisfying closure and associative which is called semi group.

**Monoid:** A set satisfying closure , associative ,identity which is called Monoid.

**Oder of the Group**

Total number of element in a Group is called order Group

**Example:**

1. (N ,+) ,(E,.) are semi group.
2. (N,.) ,(Z,.) ,(Q,.) ,(R,.) (C,.) are monoid.

3.(Z,+) ,(Q,+) ,(R,+) ,(C,+) ,(Q-{0} ,.) ,(R-{0},.) ,(C-{0} ,.)are abelian group.

4.The set of all unimodular complex numbers under multiplication of complex numbers is a group.

5.The set of all m×n matrices under the addition of matrices is an abelian group.

6. The set of all n×n non-singular matrices under the multiplication of matrices is finite abelian group.

7.$4^{th}$ root of unity {1,-1,i,-i} is an abelian group under multiplication.

8. {1,-1} is a Group under multiplication

8.$3^{rd}$ root or unity {1,$\omega$,$\omega^2$} is an abelian group under multiplication.

9.The set of all nth root of unitys under multiplication of complex number is an abelian group.

10. $Z_5$ = {[0],[1],[2],[3],[4]} ,the set of all residue of integer modulo 5 under addition modulo 5 is an abelian Group.

11.$z_p$ is an abelian ,where p is prime number.

12.(E,+) is an abelian group, where E is the set of even numbers.

13.G ={$2^n$ /n$\epsilon$Z} is a group under multiplication.[ identity $2^0$ ,inverse of $2^n$ is $2^{-n}$ ]

14.G ={$f_1$ ,$f_2$,$f_3$ ,$f_4$} defined by $f_1$(z) =z,$f_2$(z) =-z ,$f_3$(z) =$\frac{1}{z}$ ,$f_4$(z) =-$\frac{1}{z}$ is an abelian group under composition of mapping.[ f1 is identity, inverse of f1 is $f_1$, inverse of $f_2$ is $f_2$, inverse of $f_3$ is f inverse of f4 is f4,]

15.(Z,*) is an finite abelian group where * is defined as a*b =a+b+2

Identity e =-2 ,Inverse of a is,$a^{-1}$=-a-4

16. set of all matrices of the form { $\begin{pmatrix} x & x \\ x & x \end{pmatrix}$ /x$\epsilon$R-{0} } is a group under matrix multiplication.

Identity E = $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$,Inverse of A = $\begin{pmatrix} x & x \\ x & x \end{pmatrix}$is,$A^{-1}$=$\begin{pmatrix} \frac{1}{4x} & \frac{1}{4x} \\ \frac{1}{4x} & \frac{1}{4x} \end{pmatrix}$,

17.G be the set of all rational number except 1 and * be defined on G by a *b =a+b-ab,then (G,*) is an abelian group. { Identity e =0 ,Inverse of a is,$a^{-1}$=$\frac{a}{a-1}$ }

18. Let G = $S_3$ be the set of one-one mappings of the set {$x_1$,$x_2$,$x_3$} onto itself ,It is a Group of order 6 under the product

19. Let n be a integer. G ={ $a^i$ / $i$ = 0,1,2, .... $(n-1)$} is a group under multiplication

20. Let G be the set of all 2× 2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where a,b,c,d are real numbers,such that ad-bc≠ 0 is a Group under multiplication.

21. Let G be the set of all 2× 2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where a,b,c,d are real numbers,such that ad-bc = 1 is a Group under multiplication.

22. Let G be the set of all 2× 2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where a,b,c,d are real numbers, not both zero,such that $a^2 + b^2 \neq 0$ is an abelian Group under multiplication.

23. Let G be the set of all 2× 2 matrices $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ where a,b,c,d are real numbers, not both zero,such that $ad \neq 0$ is an abelian Group under multiplication.

24. Let G be the set of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where a, b, c, d are integers modulo 2, such that ad - bc ≠0 Using matrix multiplications as the operation in G ,then G is a group of order 6.

**Solution:**

In the first row of any matrix belonging to G, each entrycould be 0 or 1 in $Z_2$, but (0, 0) should be extracted since ad - bc ≠ 0Hence we have $2^2$ - 1 different choices for the first row. The second row is not a multiple of the first row. Hence G has $(2^2 - 1)2$ elements, namely 6.

$$G = \{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \}$$

25. Let G be the set of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where a, b, c, d are integers modulo 3, such that ad - bc ≠0 Using matrix multiplications as the operation in G ,then G is a group of order 48.

**Solution:**

In the first row of any matrix belonging to G, each entrycould be 0 or 1 in $Z_2$, but (0, 0) should be extracted since ad - bc ≠ 0Hence we have $3\times 3 - 1$ different choices for the first row. The second row is not a multiple of the first row. Second row (3×3 )-3 possibilities Hence the number of elements in D is 8×6 = 48

**Properties:**

If G be the group,

1.The identity element of G is unique.

2.Every a$\epsilon$G has aunique invers in G.

3.For every a$\epsilon$G, $(a^{-1})^{-1}$=a.

4.For all a,b $\epsilon$G,$(a * b)^{-1} = b^{-1} * a^{-1}$

5.For all a,b $\epsilon$G,

(i)a*b =a*c ⇒ b =c [left cancellation law]

(ii)b*a =c*c ⇒ b =c [right cancellation law]

6. For all a,b $\epsilon$G,the equation a*x =b and y*a =b have unique solution for x and y in G,the solutions are x = $a^{-1}$*b and y = b*$a^{-1}$.

7.$(a * b)^2 = a^2 * b^2$ for all a,b $\epsilon$G iff G is an abelian group.

8.If every element of a group G is its own inverse, then G is an abelian .

9.every group of order FOUR is an abelian.

9. If G is an group in which $(a * b)^k = a^k * b^k$ for all three consecutive integers k and for all a,b $\epsilon$G,then G is an abelian .

10. If the Group G has three element ,it must be abelian.

11.A group having 4 or lessthan 4 elements is an abelian group.

12.If G is a finite group,then there exists a positive integer N such that $a^N$ =e for all a$\epsilon$G.

13. If G is a group of even order, prove that it has an element a ≠ e satisfying $a^2 = e$

14. If G is group of prime order,Than G is an abelian **(TRB-2004)**

**SUBGROUP:**

A non empty subset H of agroup G is called a subgroup of G if H itself form a group under the same operation defined on G.

**Example**

1.(E,+) is a subgroup of (Z,+)

2.{1,-1} is subgroup of {1,-1,i,-i}

3.(Z,+) is subgroup of (Q,+)

4. (Z,+) is subgroup of (R,+)

5. Zn is a subgroup of Z under addition,where n$\in Z$

6. Let G be the Group of integers under addition,H the subset consisting of the multiples of 5 ,then H is a subgroup of G.

7. Let G be the Group of nonzero real numbers under multiplication, and let H be the sub set of positive rational numbers, then H is a subgroup of G.

8. Let a and b be integers.

Prove that the subset aZ + bZ = {ak + bl / l, k ∈ Z } is a subgroup of Z

9. Let G be the set of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with ad - bc ≠0 Using matrix multiplication

10. Let H = {$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in G$ / ad ≠ 0 } ,then H is a subgroup of G

11. Let G be the Group of all nonzero complex numbers a+ib ( a,b real, not both zero ) under multiplication ,and H = { a+ib∈G / $a^2 + b^2 = 1$ } is a subgroup of G

**The center of a Group**

The center of a group G is Z(G) = {x ∈ G / ax = xa for all a ∈G}, Then Z(G) is a subgroup of G.

**Normalizer (or) centralizer**

N(a) = { x∈ $G\backslash ax = xa$ } is a subgroup of G and it is called Normalizer of G. N(a) is a subgroup of G

The center of G is the intersection of all the centralizer subgroups of G.

**Theorem**

- A non empty subset H of a group G is a subset of G ⇔ (i) a,b ϵH⇒a*b ϵH (ii)a ϵH ⇒ $a^{-1}$ ϵH
- A non empty subset H of a group G is a subset of G ⇔ a,b ϵH⇒a*$b^{-1}$ ϵ H
- If H is a non empty subset finite subset of a group G and H is closed under the product in G ,than H is a subgroup of G.
- If H and K are any two non empty subgroup of G,than $(H * K)^{-1}$ =$K^{-1} * H^{-1}$
- A non empty subset H of a group G is a subset of G,H is asubgroup of G iff HH =H and $H^{-1} = H$
- If H and K are subgroup of G, HK is subgroup of G iff HK =KH
- If H and K are subgroup of G,than H∩ $K$ is also a subgroup of G
- Intersection of any number of subgroups of G is a subgroup of G
- H∪ $K$ is a subgroup of G iff H⊂ $K$ $or$ $K \subset H$
- If H and K are subgroup of abelian group G, HK is subgroup of G.
- If H and K are two subgroup of a finite group G, and H⊆ $K$ $Than [G:H] = [G:K][K:H]$
- If H and K are two finite subgroup of a group G and if O(H) ,O(K) are relatively prime,

  thanH∩ $K = \{e\}$
- If H and K are finite subgroup of G ,Than o(HK) = $\frac{o(H)o(k)}{o(H \cap K)}$
- If H and K are subgroup of a finite group G and o(H)> $\sqrt{o(G)}$ ,o(K) )> $\sqrt{o(G)}$ than H∩ $K \neq \{e\}$
- aHa$^{-1}$ = {aha$^{-1}$\h∈ $H$} is a sub group of G

# QUESTIONS FOR FIRST CLASS WITH ANSWER

1. Which of the following is not a Group

   (a) (Z,+) (b) (Q,+) **(c) (R, .)** (d) (Q-{0} ,.)

2. (Z,*) is an finite abelian group where * is defined as a*b =a+b+2,then inverse element of a$\in G$ is,

   (a) a-4 **(b) –a-4** (c)-a+4 (d) –a-2

3. G = { $\begin{pmatrix} a & a \\ a & a \end{pmatrix}$: $a \in R - \{0\}$ } is a Group under multiplication,then inverse of $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$ $is$

   (a) $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ (b) $\begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}$ **(c)** $\begin{pmatrix} \mathbf{\frac{1}{8}} & \mathbf{\frac{1}{8}} \\ \mathbf{\frac{1}{8}} & \mathbf{\frac{1}{8}} \end{pmatrix}$ (d) $\begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix}$

4. Let G = $\{f_1, f_2, f_3, f_4\}$ is a Group under composition of the functions,then invers of $f_3$is,where$f_1(z) = z, f_2(z) = -z, f_3(z) = \frac{1}{z}, f_4(z) = -\frac{1}{z}$

   (a) $f_1$ (b) $f_2$ **(c)** $\boldsymbol{f_3}$ (d) $f_4$

5. Let G be the set of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where a, b, c, d are integers modulo 3, such that ad - bc ≠0 Using matrix multiplications as the operation in G ,then G is a group of order

   (a) **48**. (b) 18 (c) 6 (d) 24

6. If G is a finite group of n, then for every a$\in G$, we have

   **(a)** $\boldsymbol{a^n = e}$ (b) $a^n = a^{-1}$ (c) $a^n = a$ (d) None of these

7. {1,-1} is a sub group of the group under multiplication

   (a) {1,I,-i} **(b) {1,-1,i,-i}** (c) {1,0,-1,i} (d) {-1,I,-I }

8. If $e_1$ and $e_2$ are two identity element of group G, then

   **(a)** $\boldsymbol{e_1 = e_2}$ (b) $e_1 \neq e_2$ (c) $e_1$ = c $e_2$ (d)None of these

9. If G is a group ,then for all a,b $\in G$

   (a) $(ab)^{-1} = a^{-1}b^{-1}$ **(b)** $\boldsymbol{(ab)^{-1} = b^{-1}a^{-1}}$ (c) $(ab)^{-1} = ab$ (d) $(ab)^{-1} = ba$

10. If G is a group , such that $(ab)^n = a^n b^n$ for three consecutive integers n for all a,b$\in G$,then G is

    **(a) abelian** (b) non-abelian (c) cyclic (d) additive group

11.Let G be the set of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where a, b, c, d are integers modulo 2, such that ad - bc ≠0 Using matrix multiplications as the operation in G ,then G is a group of order is,

**(a)** 2 (b) 3 (c) 4 **(d) 6**

12.If $H_1$ and $H_2$ are two subgroup of G, then folloeing is also a group of G

**(a) $H_1 \cap H_2$** (b) $H_1 \cup H_2$ (c) $H_1H_2$ (d) None of these

13.If axa = b,then x is equal to

(a) $a^{-1}b$ (b) $a^{-1}b^{-1}$ (c) $a^{-1}b^{-1}b^{-1}$ **(d) $a^{-1}ba^{-1}$**

14.If G is a Group ,for a∈ $G$,N(a) is the normalize of a, then for all x∈ N(a)

**(a) xa=ax** (b) xa = e (c) ax = e (d) xa≠ax

15.If G is a group such that $a^2 = e \; for \; all \; a \in G, then \; G \; is$

**(a) abelian group** (b)non abelian group (c) ring (d) field

16.If G is a group and a∈ $G$,such that $a^2 = a$ ,then 'a' is equal to

**(a) identity element** (b) inverse (c) zero element (d) None of these

17.If H,K are two subgroup of G,then Hk is a subgroup of G ,iff

(a) HK =1 **(b) HK =KH** (c) HK =$H^{-1}K^{-1}$ (d) None of these

18.For all a,b ϵG,the equation a*x =b and y*a =b have unique solution for x and y in G,the solutions are

(a) x = a*b and y = ba (b) x = a$b^{-1}$ and y = $a^{-1*}$b

**(c) x = $a^{-1*}$b and y = b*$a^{-1}$.** (d)x = b*$a^{-1}$ and y = $a^{-1*}$b

19.If H is a subgroup of G,then which of the following correct

(i) $H^{-1}$ = H (ii) h∈ $H \Rightarrow h^{-1} \in H$ (iii) $H^{-1} \neq H$ (iv) $h^{-1} \in H^{-1}$ then h∈ $H$

(a) (i),(ii) (b) (ii), (iii), (iv) **(c) (i),(ii), (iv)** (d) (i), (iv)

20.If H and K are two finite subgroup with order 6 and 5 of a group G ,then O(H∩ $K$) is,

(a) **1** (b) 6 (c) 5 (d) 30

**Continue class –II follows . . . .**

**-Material Available with Question papers-**

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